4. General Second Order Strongly Elliptic Operators

4.1. Set-Up. Let $\Omega \subseteq \mathbb{R}^n$ be bounded and open. We consider operators of the form

$$A = -\sum_{j,k=1}^{n} a^{jk}(x)\partial_{x_j}\partial_{x_k} + \sum b^j(x)\partial_{x_j} + c(x), \tag{1}$$

where $a^{j,k}, b^j$ and c are continuous, real-valued functions, $a^{jk} = a^{kj}$ and

$$c_a|\xi|^2 \le \sum a^{jk}(x)\xi_j\xi_k \le C_a|\xi|^2 \tag{2}$$

with suitable constants c and C.

We define the forms

$$q^A_{D/N}(u,v) = \int_\Omega \sum_{j,k} a^{jk}(x)\partial_{x_k}u(x)\partial_{x_k}\overline{v}(x) + \sum_j b^j(x)u(x)\overline{v}(x) + c(x)u(x)\overline{v}(x)\,dx \tag{3}$$

with domain $W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega)$ for q^D_D and $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ for q^D_N. By Theorem 2.4 they furnish self-adjoint, positive operators $A_{D/N}$.

In a first step we will also assume that all coefficients are constant. Since $a = (a^{jk})$ is real-symmetric, we can write

$$a = s^* ds,$$

where s is an orthogonal matrix and d is diagonal.

We start with the case of a diagonal operator

$$D = -\sum_{k=1}^{n} d_k^2 \partial^2_{x_k}$$

and denote by $D_{D/N}$ the operators obtained from the corresponding restriction of the form (3).

4.2. Theorem. *Let Ω be a bounded, contented open set in $\mathbb{R}^n$. Then*

$$N_{D_{D/N}}(\lambda) = \prod_{k=1}^{n} d_k^{-1}\ N_{(-\Delta)_{D/N}}(\lambda).$$

Proof. Fix $r > 0$. Following Theorem 3.1 we see that on the interval $I_{rd} = \prod[0, rd_k]$ the functions

$$\begin{aligned} s_j(x) &= \prod_{k=1}^{n} \sqrt{\frac{2}{rd_k}} \sin\left(j_k x_k \frac{\pi}{rd_k}\right), \quad j \in \mathbb{N}^n \text{ and} \\ c_j(x) &= \prod_{k=1}^{n} \sqrt{\frac{2}{rd_k}} \cos\left(j_k x_k \frac{\pi}{rd_k}\right), \quad j \in \mathbb{N}_0^n \end{aligned}$$

form a complete set of eigenfunctions for the operators D_D and D_N, respectively. We have

$$\begin{aligned} D_D s_j &= \sum_{k=1}^{n} \frac{j_k^2\pi^2}{r^2} s_j, j \in \mathbb{N}^n \\ D_N c_j &= \sum_{k=1}^{n} \frac{j_k^2\pi^2}{r^2} c_j, j \in \mathbb{N}_0^n. \end{aligned}$$

For the counting functions we therefore obtain:

$$N_{D_{D/N}}(\lambda; I_{rd}) = \left\{ j : \sum_k \frac{j_k^2\pi^2}{r^2} \le \lambda \right\} = \left\{ j : \|j\| \le \frac{r}{\pi}\sqrt{\lambda} \right\} = N_{(-\Delta)_{D/N}}(\lambda; I_r)$$

Here, $N_{(-\Delta)_{D/N}}(\lambda; I_r)$ is the counting function of the Dirichlet/Neumann Laplacian on the cube $I_r = [0, r]^n$, which, as we know, satisfies

$$N_{(-\Delta)_{D/N}}(\lambda; I_{rd}) = \frac{1}{(2\pi)^n}\,\mathrm{vol}(B(0,1))r^n\lambda^{n/2} + O((r^2\lambda)^{(n-1)/2})$$

For fixed $m \in \mathbb{N}$ we next tile $\mathbb{R}^n$ by the half-open intervals

$$\Big[\frac{z_1 d_1}{2^m}, \frac{(z_1+1)d_1}{2^m}\Big) \times \ldots \times \Big[\frac{z_n d_n}{2^m}, \frac{(z_n+1)d_1}{2^m}\Big), \qquad z = (z_1, \ldots, z_n) \in \mathbb{Z}^n.$$

Just as for the Dirichlet/Neumann Laplacian we can then apply the Dirichlet-Neumann bracketing. The difference is that we have replaced the cubes of side length 2^{-m} by intervals of side lengths $2^{-m}d_k$, $k = 1, \ldots, n$. Their volume differs by a factor $\prod d_k$ from the cubes' volume. The number of intervals of the form I_d needed to cover Ω from the interior/exterior (in the sense of the of the sets $\Omega_m^{\mp}$ in the proof of Theorem 3.14) therefore is $\prod d_k^{-1}$ times the number of cubes. This shows the assertion. □

4.3. Coordinate transforms. Let $T : V \to U$ be a diffeomorphism of open sets in $\mathbb{R}^n$ and let A be a (differential) operator defined on, say $C_c^\infty(V)$. Then we obtain a (differential) operator T_*A on $C_c^\infty(U)$ by

$$(T_*Au)(x) = A(U \circ T)(T^{-1}x).$$

For later use we compute the first and second order derivatives, assuming for the moment that T is linear, writing $T = (t_{lm})$:

$$\begin{aligned}
\partial_{x_k}(u\circ T)(x) &= \partial_{x_k}\Big(u\Big(\Big(\sum_m t_{lm}x_m\Big)_l\Big)\Big)\\
&= \sum_{p=1}^n (\partial_{y_p}u)\Big(\Big(\sum_m t_{lm}x_m\Big)_l\Big)t_{pk} \quad \text{and}\\
\partial_{x_j}\partial_{x_k}(u\circ T)(x) &= \partial_{x_j}\Big(\sum_{p=1}^n (\partial_{y_p}u)\Big(\Big(\sum_m t_{lm}x_m\Big)_l\Big)t_{pk}\Big)\\
&= \sum_{p,q=1}^n (\partial_{y_q}\partial_{y_p}u)\Big(\Big(\sum_m t_{lm}x_m\Big)_l\Big)t_{qj}t_{pk}\Big)
\end{aligned}$$

Thus

$$\partial_{x_j}\partial_{x_k}(u\circ T)(T^{-1}x) = \sum_{p,q=1}^n (\partial_{y_q}\partial_{y_p}u)(x)t_{qj}t_{pk}$$

Writing h for the Hessian matrix $(h_{qp})_{qp} = (\partial_{y_q}\partial_{y_p})_{qp}$, we obtain

$$T_*(\partial_{x_j}\partial_{x_k}) = \Big(\sum_{q=1}^n\sum_{p=1}^n h_{qp}t_{qj}t_{pk}\Big)_{jk} = (T^*hT)_{jk}.$$

4.4. Example. We consider the principal part A^H of the operator A in 4.1 with constant coefficients:

$$A^H = -\sum_{jk} a^{jk}\partial_{x_j}\partial_{x_k}.$$

As in 4.1(3) we assume that $a = s^*ds$ for an orthogonal matrix s and a diagonal matrix $d = \mathrm{diag}(d_1^2, \ldots, d_n^2)$. We also suppose that the boundary of Ω is C^2.

We derive from (4.3) that

$$\begin{aligned} s_*A^H &= -\sum_{jk} a^{jk}(s^*hs)_{jk} = -\sum_{jk} a^{jk}(s^*hs)_{jk} \\ &= -\operatorname{Tr}(as^*hs) = -\operatorname{Tr}(sas^*h) = \operatorname{Tr}(dh) \\ = &= -\sum_k d_k^2\partial_{x_k}^2 = D \end{aligned}$$

with the operator D defined in 4.1.
By $A^H_{D/N}$ we denote the self-adjoint, positive operators obtained from the forms

$$q_{D/N}(u,v) = \sum_{jk}\int_\Omega a^{jk}\partial_{x_j}u\partial_{x_k}\overline{v}\,dx$$

with domains $W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)\times W^{1,2}(\Omega)$. respectively. These operators have compact resolvent and therefore their spectrum consists of eigenvalues $0\le\lambda_k\to+\infty$. Since the boundary is C^2, the operators A^H_D and A^H_N act on their domains as the operator A^H.
Now we make the following observation: Suppose v is an eigenfunction for the eigenvalue λ of A^H. Then $u=v\circ s^{-1}$ is an eigenfunction for the eigenvalue λ of s_*A^H:

$$(s_*A^H)u(x) = A^H(u\circ s)(s^{-1}(x)) = (A^Hv)(s^{-1}(x)) = \lambda v(s^{-1}(x)) = \lambda u(x).$$

Conversely, if u is an eigenfunction for the eigenvalue λ of s_*A^H, then $v=u\circ s$ is an eigenfunction for A^H. Hence the spectra and also the counting function for A^H and s_*A^H coincide. We therefore obtain

$$N_{A^H_{D/N}}(\lambda) = N_{s_*A^H_{D/N}}(\lambda) = N_{D_{D/N}}(\lambda) = \frac{1}{\prod d_k}N_{-\Delta_{D/N}}(\lambda) = \frac{1}{\sqrt{\det a}}N_{-\Delta_{D/N}}(\lambda).$$

4.5. The constant coefficient case. Assume Weyl asymptotics hold for the Dirchlet and the Neumann Laplacian on Ω. For the Dirichlet and Neumann realizations $A_{D/N}$ of the operator A in 4.1 with constant coefficients we have.

$$N_{A_{D/N}}(\lambda) = (\det a)^{-1/2}\frac{\operatorname{vol}\Omega\operatorname{vol}B(0,1)}{(2\pi)^n}\lambda^{n/2} + o(\lambda^{n/2}).$$

Proof. Consider the forms

$$q_{D/N}(u,v) = \int_\Omega \sum_{jk} a^{jk}\partial_{x_j}u\partial_{x_k}\overline{v} + \sum_{j=1}^n b^j\partial_{x_j}u\overline{v} + cu\overline{v}\,dx$$

with domains $W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)\times W^{1,2}(\Omega)$, respectively. These forms are semi-bounded: For every j and every $\varepsilon>0$ there exists a C_ε such that

$$b^j\partial_{x_j}u\overline{v} \le |b^j||\partial_{x_j}u\overline{v}| \le |b^j|\varepsilon|\partial_{x_j}u|^2 + C_\varepsilon|v|^2.$$

By assumption $\sum_{j,k}a^{jk}\,\xi_j\xi_k\ge c_a|\xi|^2$ for some constant $c_a>0$. We deduce that, for every $\varepsilon>0$, we find a C_ε, depending on ε, b and c with

$$q_{D/N}(u,u) \ge (c_a-\varepsilon)\|\nabla u\|^2_{L^2(\Omega)} - C_\varepsilon\|u\|^2_{L^2(\Omega)}.$$

It is also closed, so that we obtain the self-adjoint operators $A_{D/N}$ from Theorem 2.4. They have compact resolvents and correspondingly, a discrete set of real eigenvalues tending to $+\infty$.

In order to determine the asymptotic of the respective counting functions of A from those of A^H, we compare the forms. The estimate

$$|\sum_{j=1}^n \int b^j \partial_{x_j} u\overline{u} + cu\overline{u}\, dx| \leq \varepsilon \|\nabla u\|_{L^2(\Omega)} + C_\varepsilon \|u\|^2$$

for arbitrary $\varepsilon > 0$ and suitable C_ε implies that, also for arbitrary $\varepsilon > 0$ and suitable C_ε,

$$\begin{aligned} q_{D/N}(u,u) &\leq (1+\varepsilon) q^{A^H}_{D/N}(u,u) + C_\varepsilon \|u\|_{L^2(\Omega)} \quad \text{and} \\ q_{D/N}(u,u) &\geq (1-\varepsilon) q^{A^H}_{D/N}(u,u) - C_\varepsilon \|u\|_{L^2(\Omega)}. \end{aligned}$$

Since the forms $q_{D/N}$ and $q^{A^H}_{D/N}$ have the same domains, we infer from the minimax principle that

$$\lambda_k^{A_{D/N}} \leq (1+\varepsilon)\lambda_k^{A^H_{D/N}} + C_\varepsilon \quad \text{and} \tag{1}$$

$$\lambda_k^{A_{D/N}} \geq (1-\varepsilon)\lambda_k^{A^H_{D/N}} - C_\varepsilon. \tag{2}$$

Hence $\lambda_k^{A^H_{D/N}} \leq \lambda$ implies that $(1-\varepsilon)\lambda_k^{A^H_{D/N}} - C_\varepsilon. \leq \lambda$ which in turn shows that $\lambda_k^{A^H_{D/N}} \leq \frac{1}{1-\varepsilon}\lambda + \frac{C_\varepsilon}{1-\varepsilon}$. We conclude from (2) that

$$\begin{aligned} N_{A_{D/N}}(\lambda) &\leq N_{A_{D/N}}(\frac{1}{1-\varepsilon}\lambda + \frac{C_\varepsilon}{1-\varepsilon}) \\ &= (\det a)^{-1/2} N_{-\Delta_{D/N}}\Big(\frac{1}{1-\varepsilon}\lambda + \frac{C_\varepsilon}{1-\varepsilon}\Big) + O\Big(\Big(\frac{1}{1-\varepsilon}\lambda + \frac{C_\varepsilon}{1-\varepsilon}\Big)^{\frac{n-1}{2}}\Big) \end{aligned}$$

Similarly, (1) implies that

$$\begin{aligned} N_{A_{D/N}}(\lambda) &\geq N_{A_{D/N}}(\frac{1}{1+\varepsilon}\lambda - \frac{C_\varepsilon}{1+\varepsilon}) \\ &= (\det a)^{-1/2} N_{-\Delta_{D/N}}\Big(\frac{1}{1+\varepsilon}\lambda - \frac{C_\varepsilon}{1+\varepsilon}\Big) + O\Big(\Big(\frac{1}{1+\varepsilon}\lambda - \frac{C_\varepsilon}{1+\varepsilon}\Big)^{\frac{n-1}{2}}\Big) \end{aligned}$$

This implies the desired estimate: In fact, we know that $C_\varepsilon \leq C/\varepsilon$ for suitable C. Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a function satisfying

$$f(\lambda) \geq (1-\varepsilon)\lambda^{n/2} - \frac{C}{\varepsilon} o(\lambda^{n/2}) \text{ and } \ f(\lambda) \leq (1+\varepsilon)\lambda^{n/2} + \frac{C}{\varepsilon} o(\lambda^{n/2})$$

Suppose $\liminf_{\lambda\to\infty} f(\lambda)\lambda^{-n/2} - 1 < -\delta$ for some $\delta > 0$. Choosing $\varepsilon = \delta/2$ this would imply that for some sequence $\lambda^{(k)}$ we would have, for all k,

$$-\frac{\delta}{2} + \frac{2C}{\delta}(\lambda^{(k)})^{-1/2} < -\delta,$$

which is not possible. In the same way we can treat $\limsup_{\lambda\to\infty} f(\lambda)\lambda^{-n/2} - 1$. □

4.6. Variable coefficients, Dirichlet boundary conditions. Let Ω be open and contented. For the Dirichlet realization A_D of the operator A in 4.1 with variable coefficients we then have.

$$N_{A_D}(\lambda) = \frac{\operatorname{vol}_g \Omega \operatorname{vol} B(0,1)}{(2\pi)^n} \lambda^{n/2} + o(\lambda^{n/2}),$$

where $\operatorname{vol}_g \Omega$ is the volume of Ω measured in the Riemannian metric $g = a^{-1}$.

Proof. For fixed $m \in \mathbb{N}$ we next tile $\mathbb{R}^n$ by the half-open intervals

$$I_z = \Big[\frac{z_1 d_1}{2^m}, \frac{(z_1+1)d_1}{2^m}\Big) \times \Big[\frac{z_n d_n}{2^m}, \frac{(z_n+1)d_1}{2^m}\Big), \quad z = (z_1, \ldots, z_n) \in \mathbb{Z}^n.$$

On each interval I_z consider the form

$$q_D(u,v) = \int_{I_z} \sum_{jk} a^{jk} \partial_{x_j} u \partial_{x_k} \overline{v} + \sum_{j=1}^{n} b^j \partial_{x_j} u \overline{v} + cu\overline{v}\, dx$$

with domain $W_0^{1,2}(I_z) \times W_0^{1,2}(I_z)$. This form is closed and semi-bounded by the same arguments as in the proof of 4.5. It defines a selfadjoint operator A_D with discrete real spectrum consisting of eigenvalues $\lambda_k^{A_D} j$ tending to ∞.

Denote by x_z the midpoint of I_z. We can then compare the forms q_D with the constant coefficient forms

$$q_D^z(u,v) = \int_{I_z} \sum_{jk} a^{jk}(x_z) \partial_{x_j} u \partial_{x_k} \overline{v} + \sum_{j=1}^{n} b^j(x_z) \partial_{x_j} u \overline{v} + c(x_z) u\overline{v}\, dx$$

Let $\varepsilon > 0$ be given. By taking m large (and hence the diameter of the intervals small) we can achieve that, with suitable $C_\varepsilon > 0$,

$$\begin{aligned} q_D(u,u) &\leq (1+\varepsilon) q_D^z(u,u) + C_\varepsilon \|u\|_{L^2(\Omega)} \quad \text{and} \\ q_D(u,u) &\geq (1-\varepsilon) q_D^z(u,u) - C_\varepsilon \|u\|_{L^2(\Omega)}. \end{aligned}$$

As in the proof of 4.5 we conclude that the counting functions on each I_z satisfy

$$\begin{aligned} N_{A_D}(\lambda) &\geq (1-\varepsilon)(\det a_z)^{-1/2}(2\pi)^{-n} \operatorname{vol} I_z \operatorname{vol} B(0,1) \lambda^{n/2} - C_\varepsilon \cdot o(\lambda^{n/2}) \quad \text{and} \\ N_{A_D}(\lambda) &\leq (1+\varepsilon)(\det a_z)^{-1/2}(2\pi)^{-n} \operatorname{vol} I_z \operatorname{vol} B(0,1) \lambda^{n/2} + C_\varepsilon \cdot o(\lambda^{n/2}) \end{aligned}$$

where $a_c = (a^{jk}(x_z))_{jk}$. Dirichlet-Neumann bracketing and taking the limit as $m \to \infty$ then implies that

$$N_{A_D}(\lambda) = \int_\Omega (\det a(x))^{-1/2}\, dx \frac{volB(0,1)}{(2\pi)^n} \lambda^{n/2} + o(\lambda^{n/2}).$$

Finally we note that

$$\int_\Omega (\det a(x))^{-1/2}\, dx = \int_\Omega \sqrt{\det a(x)^{-1}}\, dx = \operatorname{vol}_g(\Omega)$$

for the Riemannian metric $g = a^{-1}$. □

4.7. Remark. Second order operators on manifolds. Let Ω be a smooth manifold of dimension n, and let A be a strongly elliptic second order operator that locally is of the form in 4.1.

According to Whitehead [12] every smooth manifold admits a triangulation. For each simplicial set we can consider the corresponding operator on a simplex in $\mathbb{R}^n$. On the simplex, we can derive Weyl asymptotics as in 4.6 for the local operator.

I expect that, via Dirichlet-Neumann bracketing we then obtain the Weyl asymptotics on the manifold.