3. Weyl's Formula for the Asymptotics of the Eigenvalues of the Dirichlet Laplacian

In the sequel let Ω be a bounded domain in $\mathbb{R}^n$. Weyl's formula states that $N_D(\lambda) = \sum_{\lambda_j<\lambda} 1$, i.e. the number of eigenvalues of the Dirichlet Laplacian that are $< \lambda$ satisfies

$$N_D(\lambda) \sim (2\pi)^{-n} \operatorname{vol}(B(0,1))) \operatorname{vol}(\Omega)\lambda^{n/2}.$$

3.a. **The Dirichlet and Neumann Laplacian on Cubes.**

3.1. Theorem. *On the interval $[0,\pi]^n \subseteq \mathbb{R}^n$ we consider the functions*

$$s_j : [0,\pi]^n \to \mathbb{R}, \quad s_j(t) = \left(\frac{2}{\pi}\right)^{n/2} \sin(j_1x_1)\cdots\sin(j_nx_n)$$

for $(j_1,\dots j_n) \in \mathbb{N}^n$. Then $\{s_j : j \in \mathbb{N}^n\}$ is an orthonormal basis of $L^2([0,\pi]^n)$ consisting of eigenfunctions of $-\Delta$ with the Dirichlet boundary condition; the corresponding eigenvalue is $|j|^2$.
Similarly, the functions $(\frac{2}{\pi})^{n/2}\cos(j_1x_1)\cdots\cos(j_nx_n)$, $j \in \mathbb{N}_0^n$ furnish an orthonormal basis of L^2 consisting of eigenfunctions of Δ with Neumann boundary condition.
If instead of the cube $[0,\pi]^n$ we have $[0,a]^n$, then we can choose the functions

$$\left(\frac{2}{a}\right)^{n/2} \sin(\frac{j_1\pi}{a}x_1)\cdots\sin(\frac{j_n\pi}{a}x_n), \quad j \in \mathbb{N}^n.$$

Similarly, for the Neumann Laplacian we have the system

$$\left(\frac{2}{a}\right)^{n/2} \cos(\frac{j_1\pi}{a}x_1)\cdots\cos(\frac{j_n\pi}{a}x_n), \quad j \in \mathbb{N}_0^n.$$

Proof. First consider the one-dimensional Dirichlet case. The problem $u'' = -\lambda^2 u$, $u(0) = u(\pi) = 0$ only has a solution when $\lambda = j$ for some $j \in \mathbb{N}$, namely $\sin(jx)$. The set $\{\sqrt{2/\pi}\sin(jx) : j \in \mathbb{N}\}$ clearly is orthonormal. It remains to check that is complete. To see this, we first note that $-\Delta_D$ is selfadjoint and invertible, so its inverse is a bounded and selfadjoint operator. The inverse even is compact, since it maps $L^2([0,\pi])$ into $W_0^1([0,\pi])$ which in turn embeds compactly into $L^2([0,\pi])$ by Theorem 2.8. Hence the eigenfunctions of the inverse form an orthonormal basis for $L^2([0,\pi])$. As the eigenvalues of $-\Delta_D$ coincide with those of $(-\Delta_D)^{-1}$ this proves the 1-dimensional case.
By taking tensor products, we obtain the assertion in the general case: Since the span of $\{\sqrt{2/\pi}\sin(jx) : j \in \mathbb{N}\}$ is dense in $L^2([0,\pi])$ the span of

$$\{(2/\pi)^{n/2}\prod_{k=1}^{n}\sin(j_kx_k) : j \in \mathbb{N}^n\}$$

is dense in the algebraic tensor product $\bigotimes L^2([0,\pi])$ which in turn is dense in $L^2([0,\pi]^n)$.
The proof in the Neumann case is analogous. In order to avoid problems with the non-invertibility, we first consider the operator $I - \Delta_N$. Its spectrum is that of $-\Delta_N$, shifted by 1. So it is invertible and selfadjoint; its inverse is selfadjoint and compact. Its eigenfunctions are those of $I - \Delta_N$ which in turn are those of $-\Delta_N$. □

3.2. Proposition. *Let Ω be a cube in $\mathbb{R}^n$.*
(a) *Let $\mathscr{D}_D = \{f \in C^\infty(\Omega) : f = 0$ on $\partial\Omega\}$. Then $\mathscr{D}_D$ is an operator core for $-\Delta_D$, and, for $u \in \mathscr{D}_D$,*

$$-\Delta_D u = -\sum_{j=1}^{n}\frac{\partial^2 u}{\partial {x_j}^2}. \qquad (1)$$

(b) *Let* $\mathscr{D}_N = \{f \in C^\infty(\Omega) : \partial_\nu f = 0$ *on* $\partial\Omega\}$. *Then* $\mathscr{D}_N$ *is an operator core for* $-\Delta_N$, *and* (1) *also holds for* $u \in \mathscr{D}_N$.

Proof. (a) W.l.o.g. assume that $\Omega =]0,\pi[^n$ and let $A = -\Delta$ be the Laplacian with domain $\mathscr{D}_D$. We have to show that its closure $\overline{A} = \overline{-\Delta}$ equals $-\Delta_D$. We denote by $\{s_j : j \in \mathbb{N}^n\}$ the above orthonormal basis. The functions s_j are elements in $\mathscr{D}_D$ and - since they form an orthonormal basis - each function in $u \in L^2([0,\pi]^n)$ has the representation

$$u = \sum_{j\in\mathbb{N}^n} c_j s_j \text{ with } c_j = \langle u, s_j\rangle \text{ and } \sum |c_j|^2 = \|u\|^2.$$

Given $u = \sum c_j s_j \in L^2$, we have $Au \in L^2$ if and only if $\sum |j|^4|c_j|^2 < \infty$. Hence

$$\mathscr{D}_{\max} = \left\{\sum c_j s_j : \sum |j|^4|c_j|^2 < \infty\right\}$$

defines the maximal domain of A in $L^2([0,\pi]^n)$.

This is also the domain of the closure $\overline{A}$, since $(u_k)_{k\in\mathbb{N}}$ given by $u_k = \sum_{|j|\le k} c_j s_j$ is a sequence in $\mathscr{D}_D$ such that $u_k \to u$ in L^2 and $Au_k \to Au$ in L^2. The proof actually shows slightly more: Even the linear combinations of the s_j form a core for A.

Clearly, $\overline{A}$ is symmetric on $\mathscr{D}_{\max}$. As the domain of the adjoint $\overline{A}^*$ contains the domain of $\overline{A}$, which is maximal, $\overline{A}$ is selfadjoint.

For $u = \sum c_j s_j$ and $v = \sum d_j s_j$ in $\mathscr{D}_D \subseteq W_0^{1,2}(\Omega)$,[2] we have

$$q(u,v) = \int \langle \nabla u, \nabla v\rangle_{\mathbb{C}^n} = \sum |j|^2 c_j \overline{d}_j = \langle -\Delta u, v\rangle.$$

Hence $-\Delta_D$ coincides with $A = -\Delta$ on $\mathscr{D}_D$. By definition, the Dirichlet Laplacian then is a closed extension of A. But we saw that the domain of the closure $\overline{A}$ already is the maximal domain $\mathscr{D}_{\max}$, so both must coincide.

(b) Similar. □

3.3. Eigenvalue asymptotics for the Dirichlet Laplacian on a cube. Let us check that Weyl's law is correct in the case where Ω is the cube $[0,a]^n$. Here

$$N_D(\lambda) = \{j \in \mathbb{N}^n : \frac{|j|^2\pi^2}{a^2} \le \lambda\} = \{j \in \mathbb{N}^n : |j| \le \frac{a}{\pi}\sqrt{\lambda}\}.$$

So $N_D(\lambda)$ counts the points of the integer lattice in $\mathbb{R}^n$ with all coordinates positive which lie inside a ball of radius $\frac{a}{\pi}\sqrt{\lambda}$. This is the same as counting the unit cubes that lie within the ball intersected with $\{x \in \mathbb{R}^n : x_k \ge 0, k = 1,\dots,n\}$. The difference between this number and the volume of the ball is bounded by the number of cubes intersecting the corresponding sphere, which is $O((\frac{a}{\pi}\sqrt{\lambda})^{n-1})$. Hence

$$\begin{aligned} N_D(\lambda) &= \frac{1}{2^n}\operatorname{vol}(B(0,\frac{a}{\pi}\sqrt{\lambda}) + O((a^2\lambda)^{(n-1)/2}) = \frac{1}{2^n}\operatorname{vol}(B(0,1)(\frac{a}{\pi}\sqrt{\lambda})^n + O((a^2\lambda)^{(n-1)/2}) \\ &= \frac{1}{(2\pi)^n}\operatorname{vol}(B(0,1))a^n\lambda^{n/2} + O((a^2\lambda)^{(n-1)/2}), \end{aligned}$$

which is what Weyl predicts.

3.4. Eigenvalue asymptotics for the Neumann Laplacian on a cube. The counting function $N_N = N_N(\lambda)$ of the Neumann Laplacian on the cube $[0,a]^n$ counts the points of the integer lattice in $\mathbb{R}^n$ with all coordinates non-negative that lie inside the ball of radius $\frac{a}{\pi}\sqrt{\lambda}$. This differs from the number we counted for the Dirichlet Laplacian by the number of integer

[2] A fact needed here, which is not completely obvious, is that $\mathscr{D}_D \subset W_0^{1,2}(\Omega)$

points on the coordinate hyperplanes inside the ball. This number is $O((a^2\lambda)^{(n-1)/2})$; hence we obtain the same result as in the Dirichlet case:

$$N_N(\lambda) = \frac{1}{(2\pi)^n}\,\mathrm{vol}(B(0,1))a^n\lambda^{n/2} + O((a^2\lambda)^{(n-1)/2}).$$

3.b. **Domain decomposition.**

3.5. The form domain of an operator. Let A be a selfadjoint operator on a Hilbert space H. If A arises from a form q via the construction in Theorem 2.4, then we call the domain $Q(q)$ of the form also the *form domain of the operator* A, and one frequently writes $Q(A)$.

Conversely, suppose that there exists an orthonormal basis $\{e_j : j \in \mathbb{N}\}$ of H of eigenvalues λ_j of A, i.e.

$$Au = \sum_{j=1}^{\infty} \lambda_j \langle u, e_j\rangle e_j, \quad u \in \mathscr{D}(A).$$

Then we can define the form

$$q_A(u,v) = \sum_{j=1}^{\infty} \lambda_j \langle u, e_j\rangle \langle v, e_j\rangle$$

for u, v with the domain

$$Q = \{u \in H : \sum_{j=1}^{\infty} |\lambda_j||\langle u, e_j\rangle|^2 < \infty\}$$

Provided $\lambda_j \geq -C$ for some C, we see that the form q_A is closed and semi-bounded and generates A in the sense of 2.4.

(More generally we can work with the spectral theorem for unbounded selfadjoint operators.)

3.6. Lemma. *Let $A_j : \mathscr{D}(A_j) \subseteq H_j \to H_j$, $j = 1, 2$, be selfadjoint operators on Hilbert spaces H_1 and H_2. We define*

$$A = A_1 \oplus A_2 : \mathscr{D}(A) \subseteq H_1 \oplus H_2 \to H_1 \oplus H_2 \quad \mathscr{D}(A) = \mathscr{D}(A_1) \oplus \mathscr{D}(A_2),$$

by $A(u,v) = (A_1u, A_2v)$ for $(u,v) \in \mathscr{D}(A)$. Then

(a) *$A_1 \oplus A_2$ is self-adjoint.*

(b) *If D_1 is a core for A_1 and D_2 is a core for A_2 then $D_1 \oplus D_2$ is a core for A.*

(c) *Assume additionally that A_1 and A_2 are as in 3.5 with orthonormal bases $\{e_j(A_1) : j \in \mathbb{N}\}$ and $\{e_k(A_2) : k \in \mathbb{N}\}$ with associated eigenvalues $\lambda_j(A_1)$ and $\lambda_k(A_2)$. Then we obtain an orthonormal basis of eigenvectors of A by taking $\{e_j(A_1) \oplus 0 : j \in \mathbb{N}\} \cup \{0 \oplus e_k(A_2) : k \in \mathbb{N}\}$ and we have $Q(A) = Q(A_1) \oplus Q((A_2)$.*

(d) *In the situation of* (c) *the counting functions* (*defined by $N_B(\lambda) = \sum_{\lambda_j(B)<\lambda} 1$ for an operator B*) *satisfy*

$$N_A(\lambda) = N_{A_1}(\lambda) + N_{A_2}(\lambda).$$

Proof. All assertions are obvious. □

3.7. Proposition. *Let Ω_1 and Ω_2 be disjoint open sets. Then $L^2(\Omega_1 \cup \Omega_2) = L^2(\Omega_1) \oplus L^2(\Omega_2)$. Moreover, under this decomposition,*

$$\begin{aligned} -\Delta_D^{\Omega_1\cup\Omega_2} &= -\Delta_D^{\Omega_1} \oplus -\Delta_D^{\Omega_2} \text{ and} \\ -\Delta_N^{\Omega_1\cup\Omega_2} &= -\Delta_N^{\Omega_1} \oplus -\Delta_N^{\Omega_2}. \end{aligned}$$

Proof. Consider first the Dirichlet case. Let $f \in C_c^\infty(\Omega_1 \cup \Omega_2)$. Then $f = f_1 \oplus f_2$ with $f_j = f_{|\Omega_j} \in C_c^\infty(\Omega_j)$, similarly for $g \in C_c^\infty(\Omega_1 \cup \Omega_2)$. Then

$$\int_{\Omega_1 \cup \Omega_2} \langle \nabla f, \nabla g \rangle_{\mathbb{C}^n} dx = \int_{\Omega_1} \langle \nabla f_1, \nabla g_1 \rangle_{\mathbb{C}^n} dx + \int_{\Omega_2} \langle \nabla f_2, \nabla g_2 \rangle_{\mathbb{C}^n} dx.$$

Hence the forms agree on $C_c^\infty(\Omega_1 \cup \Omega_2) \times C_c^\infty(\Omega_1 \cup \Omega_2)$ which is dense in $W_0^{1,2}(\Omega_1 \cup \Omega_2) \times W_0^{1,2}(\Omega_1 \cup \Omega_2)$ by definition, and the equality extends to $W_0^{1,2}(\Omega_1 \cup \Omega_2) \times W_0^{1,2}(\Omega_1 \cup \Omega_2)$.

In the Neumann case we can use the corresponding argument for the functions $f, g \in W^{1,2} \cap C^\infty(\Omega_1 \cup \Omega_2)$, which is dense in $W^{1,2}(\Omega_1 \cup \Omega_2)$. □

3.8. Corollary. *Let $\Omega_1, \ldots, \Omega_N$ be pairwise disjoint open sets in $\mathbb{R}^n$. Then we have the following relations for the counting functions of the Dirichlet and Neumann Laplacian on the respective domains:*

$$\begin{aligned} N_D(\lambda; \bigcup_{j=1}^N \Omega_j) &= \sum_{j=1}^N N_D(\lambda; \Omega_j) \text{ and} \\ N_N(\lambda; \bigcup_{j=1}^N \Omega_j) &= \sum_{j=1}^N N_N(\lambda; \Omega_j). \end{aligned}$$

3.c. **The Minimax Principle for Selfadjoint Operators.** We recall the following theorem from functional analysis

3.9. Theorem. *Let A be a bounded selfadjoint operator on a complex Hilbert space, and let $m = \inf\{\langle Au, u\rangle : \|u\| = 1\}$ and $M = \sup\{\langle Au, u\rangle : \|u\| = 1\}$. Then $\boldsymbol{\sigma}(A) \subseteq [m, M]$ and both m and M belong to $\boldsymbol{\sigma}(A)$.*

In particular, if $A \geq 0$, then $\|A\| = M$.

3.10. Theorem. *Let $A \geq 0$ be a compact selfadjoint operator on an infinite-dimensional Hilbert space H. Denote by $\mu_1 \geq \mu_2, \ldots$ the decreasing sequence of eigenvalues, repeated according to their multiplicity. Then*

$$\mu_n = \inf_{\{\varphi_1, \ldots, \varphi_{n-1}\}} M_A(\varphi_1, \ldots, \varphi_{n-1}),$$

where

$$M_A(\varphi_1, \ldots, \varphi_{n-1}) = \sup\{\langle Au, u\rangle : \|u\| = 1, u \perp LH\{\varphi_1, \ldots, \varphi_{n-1}\}\}.$$

The theorem generalizes to nonpositive operators with an analogous formula for the negative eigenvalues.

Proof. Theorem 3.9 shows the assertion for $n = 1$. Moreover, let (e_j) be an orthonormal sequence of eigenvectors associated with the eigenvalues μ_j. By changing successively A on $LH\{e_1, \ldots, e_{n-1}\}$ to zero, we see that

$$\mu_n = \sup\{\langle Au, u\rangle : \|u\| = 1, u \perp LH\{e_1, \ldots, e_{n-1}\}\}.$$

To prove the theorem, we only need to show that

$$M_A(\varphi_1, \ldots, \varphi_{n-1}) \geq \mu_n$$

for arbitrary given $\varphi_1, \ldots, \varphi_{n-1}$. To this end choose $v = \sum_{j=1}^n c_j e_j$ with $1 = \|v\|^2 = \sum |c_j|^2$ and $v \perp LH\{\varphi_1, \ldots, \varphi_{n-1}\}$ (this can be done as it leads to a system of $n-1$ equations for n unknowns). Then

$$M_A(\varphi_1, \ldots, \varphi_{n-1}) \geq \langle Av, v\rangle = \sum_{j=1}^n \mu_j |c_j|^2 \geq \mu_n.$$

□

As a corollary we obtain the following theorem.

3.11. Theorem. *Let $T \geq 0$ be an unbounded selfadjoint operator on an infinite-dimensional complex Hilbert space and suppose T is invertible with compact inverse $A \geq 0$. Denote by $0 < \lambda_1 \leq \lambda_2$ the eigenvalues of T, listed according you their multiplicity. Then $\lambda_n = \mu_n^{-1}$ for the corresponding eigenvalues of A, and we see from Theorem 3.10 that*

$$\lambda_n = \sup_{\{\varphi_1,\dots,\varphi_{n-1}\}} m_T(\varphi_1,\dots,\varphi_{n-1}),$$

where

$$m_T(\varphi_1,\dots,\varphi_{n-1}) = \inf\{\langle Tu,u\rangle : u \in \mathscr{D}(T), \|u\| = 1, u \perp LH\{\varphi_1,\dots,\varphi_{n-1}\}\}.$$

Reed and Simon point out [8, Theorem XIII.2] that in the definition of m_T the condition $u \in \mathscr{D}(T)$ can be replaced by $u \in Q(T)$.

There are much more general results for the spectrum of semi-bounded selfadjoint operators, but this is sufficient for our purposes.

3.d. **Comparison results.**

3.12. Definition. Let S and T be selfadjoint operators on the same Hilbert space. We write $0 \leq S \leq T$ if the following holds:

(i) $0 \leq \langle Su,u\rangle$ on $Q(S)$,
(ii) $0 \leq \langle Tu,u\rangle$ on $Q(T)$, and
(iii) $Q(S) \supseteq Q(T)$ with
$$0 \leq \langle Su,u\rangle \leq \langle Tu,u\rangle \text{ on } Q(T).$$

The following proposition is an immediate consequence of Theorem 3.11.

3.13. Proposition. *Assume that $0 \leq S \leq T$ and, moreover, that S and T have compact inverses C_S and C_T as in Corollary 3.11. Denote by $\lambda_j(S)$ and $\lambda_j(T)$ the eigenvalues of S and T, respectively.*

Then $\lambda_j(S) \leq \lambda_j(T)$ for all $j = 1, 2, \dots$, and the corresponding counting functions satisfy $N(\lambda, S) \geq N(\lambda, T)$.

3.14. Theorem. *Let $\Omega, \Omega', \Omega''$ be bounded domains.*
(a) If $\Omega \subseteq \Omega'$, then $0 \leq -\Delta_D^{\Omega'} \leq -\Delta_D^{\Omega}$. Via extension by zero we consider here $L^2(\Omega)$ as a subspace of $L^2(\Omega')$.
(b) $0 \leq -\Delta_N^{\Omega} \leq -\Delta_D^{\Omega}$.
(c) If Ω' and Ω'' are disjoint open subsets of Ω such that $\overline{(\Omega' \cup \Omega'')}^{\circ} = \Omega$ and $\Omega \setminus (\Omega' \cup \Omega'')$ has measure zero, then

$$\begin{aligned} 0 &\leq -\Delta_D^{\Omega} \leq -\Delta_D^{\Omega' \cup \Omega''} \\ 0 &\leq -\Delta_N^{\Omega' \cup \Omega''} \leq -\Delta_N^{\Omega} \end{aligned}$$

Proof. (a) Via extension by zero $C_c^\infty(\Omega)$ is a subspace of $C_c^\infty(\Omega')$. Therefore, the form domain on $-\Delta_D^{\Omega}$ is a subset of the form domain of $-\Delta_D^{\Omega'}$; both forms coincide on the smaller domain. By Definition 3.12 this says that $0 \leq -\Delta_D^{\Omega'} \leq -\Delta_D^{\Omega}$.
(b) The form domain of the Neumann Laplacian, namely $W^{1,2}(\Omega)$ contains that of the Dirichlet Laplacian, namely $W_0^{1,2}(\Omega)$. Both forms coincide on the smaller domain.
(c) For the Dirichlet Laplacian we obtain the assertion from (a), even without the assumption on the measure. For the Neumann case observe that for $u \in W^{1,2}(\Omega)$, we have $u_{|\Omega'} \in W^{1,2}(\Omega')$ and

$u_{|\Omega''} \in W^{1,2}(\Omega'')$. In fact, we can then identify $W^{1,2}(\Omega)$ with a subset of $W^{1,2}(\Omega' \cup \Omega'')$, since functions with the same restrictions to $\Omega' \cup \Omega''$ agree outside a set of measure zero. Moreover, this implies that, on $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$,

$$\int_{\Omega' \cup \Omega''} |\nabla u|^2 dx = \int_\Omega |\nabla u|^2 dx,$$

so that the forms coincide on the smaller domain. □

3.e. **Proof of Weyl's Theorem for Domains in $\mathbb{R}^n$.**

3.15. Covering $\mathbb{R}^n$ by standard 2^{-k} cubes. We cover $\mathbb{R}^n$ with the half-open cubes

$$\Big[\frac{a_1}{2^k}, \frac{a_1+1}{2^k}\Big) \times \ldots \times \Big[\frac{a_n}{2^k}, \frac{a_n+1}{2^k}\Big).$$

Given a bounded open subset $\Omega \subseteq \mathbb{R}^n$ we denote by

(i) $W_k^-(\Omega)$: the volume of all cubes contained in Ω,

(ii) $W_k^+(\Omega)$: the volume of all cubes intersecting Ω,

(iii) $\mathrm{vol}(\Omega)$: the volume of Ω.

We then obtain the inequalities

$$W_k^-(\Omega) \le W_{k+1}^-(\Omega) \le \mu(\Omega) \le W_{k+1}^+(\Omega) \le W_k^+(\Omega) \tag{1}$$

We call Ω *contented* (or say Ω is Jordan measurable), if

$$W_\infty^-(\Omega) := \lim W_k^-(\Omega) = \lim W_k^+(\Omega) =: W_\infty^+(\Omega).$$

Of course, in this case, the limits equal the Lebesgue measure $\mathrm{vol}(\Omega)$.

3.16. Theorem. *Let Ω be a contented bounded open set. Then*

$$\lim_{\lambda\to\infty} N_D(\lambda) = (2\pi)^{-n}\,\mathrm{vol}(B(0,1))\,\mathrm{vol}(\Omega)\lambda^{n/2} + O(\lambda^{\frac{n-1}{2}}).$$

Proof. We will show that for arbitrary k,

$$\limsup_{\lambda\to\infty} N_D(\lambda)/\lambda^{n/2} \le (2\pi)^{-n}\,\mathrm{vol}(B(0,1))W_k^+(\Omega) + O(\lambda^{-1/2}) \text{ and} \tag{1}$$

$$\liminf_{\lambda\to\infty} N_D(\lambda)/\lambda^{n/2} \ge (2\pi)^{-n}\,\mathrm{vol}(B(0,1))W_k^-(\Omega) + O(\lambda^{-1/2}) \tag{2}$$

Denote by Ω_k^- the interior of the union of all cubes contained in Ω and by Ω_k^+ the interior of the union of all cubes intersecting Ω. By $C_{k,j}^\pm$ denote the interiors of the cubes that make up $\Omega_k^\pm$. Then we conclude from Proposition 3.7 and Theorem 3.14 that

$$-\Delta_D^\Omega \overset{3.14(a)}{\le} -\Delta_D^{\Omega_k^-} \overset{3.14(a)}{\le} -\Delta_D^{\bigcup_j C_{k,j}^-} \overset{3.7}{\le} \bigoplus_j -\Delta_D^{C_{k,j}^-}. \tag{3}$$

Since, for cubes, we have already shown Weyl's theorem, we see that

$$\frac{N_D(\Omega,\lambda)}{\lambda^{n/2}} \overset{(3),\,3.6}{\ge} \sum_j \frac{N_D(C_{k,j}^-,\lambda)}{\lambda^{n/2}} = \frac{W_k^-(\Omega)\,\mathrm{vol}(B(0,1))}{(2\pi)^n} + O(\lambda^{-1/2}),$$

which is estimate (1). In an analogous way we see that

$$-\Delta_D^\Omega \overset{3.14(a)}{\ge} -\Delta_D^{\Omega_k^+} \overset{3.14(b)}{\ge} -\Delta_N^{\Omega_k^+} \overset{3.14(c)}{\ge} -\Delta_N^{\bigcup_j C_{k,j}^+} \overset{3.7}{\ge} \bigoplus_j -\Delta_N^{C_{k,j}^+}. \tag{4}$$

In view of the fact that the eigenvalues of the Neumann Laplacian on cubes also satisfy Weyl's estimate, we conclude that

$$\frac{N_D(\Omega,\lambda)}{\lambda^{n/2}} \overset{(4),\,3.6}{\leq} \sum_j \frac{N_N(C^+_{k,j},\lambda)}{\lambda^{n/2}} = \frac{W^+_k(\Omega)\,\mathrm{vol}(B(0,1))}{(2\pi)^n} + O(\lambda^{-1/2})),$$

which yields estimate (2). □

3.17. Theorem. *Let Ω be a bounded open set in $\mathbb{R}^n$ and suppose $\partial\Omega$ is sufficiently smooth Then we also have for the Neumann Laplacian*

$$\lim_{\lambda\to\infty} N_N(\lambda) = (2\pi)^{-n}\,\mathrm{vol}(B(0,1))\,\mathrm{vol}(\Omega)\lambda^{n/2} + O(\lambda^{\frac{n-1}{2}}).$$

Proof. We presently only see that the result holds for domains that are finite unions of cubes (or intervals). Using the technique established in the next section it should then hold for finite unions of diffeomorphic images of cubes. Any domain with C^1-boundary should then be admissible. □

3.18. Corollary. *Conversely, $\lambda_k \sim ck^{2/n}$.*

3.19. Remark. One might expect that one could obtain a further expansion $N(\lambda) = c_o\lambda^{n/2} + c_1\lambda^\alpha$ for $\alpha < n/2$ and a suitable coefficient c_1. In general, however, this is not possible; see Hörmander [5, p.56].