2. Operators and Forms

Let H be a complex Hilbert space with scalar product $\langle \cdot, \cdot \rangle$.

2.a. **Selfadjoint operators defined from forms.**

2.1. Definition. A *quadratic form* on H is a sesquilinear map

$$q : Q(q) \times Q(q) \to \mathbb{C}$$

where $Q(q)$ is a dense subspace of H, called the *form domain*.
We call q

- *symmetric*, if $q(u,v) = \overline{q(v,u)}$,
- *positive*, if $q(u,u) \geq 0$ and
- *semibounded*, if $q(u,u) \geq -c\|u\|^2$ for some $c \geq 0$.
- *closed*, if it is semibounded and $Q(q)$ is closed under the norm

$$\|u\|_{+1} = (q(u,u) + (c+1)\|u\|^2)^{1/2}.$$

A semibounded form is necessarily symmetric by the polarization identity.

2.2. Lemma. (a) *The norm* $\| \cdot \|_{+1}$ *is induced by the scalar product*

$$(u,v)_{+1} = q(u,v) + (c+1)\langle u, v \rangle.$$

(b) *The following are equivalent for a semibounded form:*

(i) q *is closed*
(ii) *If* (u_k) *is a sequence in* $Q(q)$ *with* $u_k \to u$ *in* H *for some* u *and* $q(u_l - u_k, u_l - u_k) \to 0$ *as* $k, l \to \infty$, *then* $u \in Q(q)$ *and* $q(u - u_k, u - u_k) \to 0$.

Proof. (a) Obvious.
(b) The condition (ii) rephrases the fact that a Cauchy sequence with respect to $\| \cdot \|_{+1}$ has a limit in $Q(q)$. □

2.3. Definition. A subset $\mathcal{D}$ of $Q(q)$ which is dense with respect to the $\| \cdot \|_{+1}$-norm is called a *form core*.

The following is the central theorem of this section:

2.4. Theorem. *If* q *is a closed semibounded form, then there exists a unique selfadjoint (unbounded) operator* A *such that*

$$q(u,v) = \langle Au, v \rangle, \quad u, v \in \mathscr{D}(A).$$

Proof. Step 1: General considerations. Without loss of generality assume q is positive, i.e. $c = 0$. Being semibounded, q is symmetric. The fact that q is closed then implies that $H_{+1} = Q(q)$ is a Hilbert space with respect to the scalar product

$$(\cdot, \cdot)_{+1} = \langle u, v \rangle + q(u,v).$$

Denote by H_{-1} the space of all bounded conjugate-linear maps $H_{+1} \to \mathbb{C}$. By Riesz' representation theorem we have an isometric isomorphism

$$J : H_{+1} \to H_{-1} \text{ via } (Ju)v = (u,v)_{+1}.$$

The map $j : u \mapsto \langle u, \cdot \rangle$ defines an embedding $H \hookrightarrow H_{-1}$: In fact,

$$|j(u)(v)| = |\langle u, v \rangle| \leq \|u\|\|v\| \leq \|u\|\|v\|_{+1}.$$

Since we have the natural injection $H_{+1} \hookrightarrow H$ we obtain the triple

$$H_{+1} \overset{\mathrm{id}}{\hookrightarrow} H \overset{j}{\hookrightarrow} H_{-1}.$$

Step 2: Definition of the auxiliary operator A_1. We let

$$\mathscr{D}(A_1) = \{u \in H_{+1} : \exists v \in H \text{ with } Ju = jv \in \operatorname{ran} j\} = J^{-1}(\operatorname{ran} j).$$

and define the action of A_1 by $A_1 u = v$, i.e. $A_1 = j^{-1}J$. This makes sense in view of the injectivity of j.
More explicitly: By definition $(ju)(v) = \langle u, v\rangle$, or $U(v) = \langle j^{-1}U, v\rangle$ for $U \in \operatorname{ran} j$, $v \in H_{+1}$. This implies that

$$\langle A_1 u, v\rangle = \langle j^{-1}Ju, v\rangle = (Ju)(v) = (u, v)_{+1}, \quad u \in \mathscr{D}(A_1), v \in H_{+1}.$$

Step 3: Density of the domain. Let us first check that the range of j is dense in H_{-1}. Suppose it were not. Then the isometric isomorphism $J : H_{+1} \to H_{-1}$ together with with Riesz' theorem shows that there exists a vector $0 \neq v \in H_{+1}$ such that

$$(J^{-1}j(u), v) = 0, u \in H.$$

By definition of J and j, we then obtain the contradiction $v = 0$, since

$$0 = (J^{-1}j(u), v)_{+1} = j(u)(v) = \langle u, v\rangle, u \in H.$$

Hence $\operatorname{ran} j$ is dense in H_{-1}. Since J is an isometry, $\mathscr{D}(A)$ is dense in H_{+1}. Now the fact that $H_{+1} = Q(q)$ is dense in H by assumption and $\|\cdot\| \le \|\cdot\|_{H_{+1}}$ implies that $\mathscr{D}(A)$ is dense in H.
Step 4: Symmetry. Let $u, v \in \mathscr{D}(A_1)$. By definition,

$$\begin{aligned}\langle A_1 u, v\rangle &= (u, v)_{+1} = \langle u, v\rangle + q(u, v) = \overline{\langle v, u\rangle + q(v, u)} \\ &= \overline{\langle A_1 v, u\rangle} = \langle u, A_1 v\rangle.\end{aligned}$$

Step 5: Selfadjointness. Let $C = J^{-1}j$. By definition, C maps H to $\mathscr{D}(A_1) = J^{-1}j(H)$ and satisfies $A_1 C = I_H$, $CA_1 = I_{\mathscr{D}(A_1)}$, i.e. A_1 is the algebraic inverse of $C : H \to \mathscr{D}(A_1)$. For $u, v \in H$, the symmetry of A_1 implies that

$$\langle Cu, v\rangle = \langle Cu, A_1 Cv\rangle = \langle A_1 Cu, Cv\rangle = \langle u, Cv\rangle.$$

The theorem of Hellinger and Toeplitz then implies that C is bounded and selfadjoint.
In order to determine the domain of A_1^*, we make two observations. The first is that $V\mathcal{G}(A_1) = \mathcal{G}(-C)$, the second that, as a consequence of the selfadjointness of C, $\mathcal{G}(C) = V(\mathcal{G}(C))^\perp$. We therefore find that

$$\mathcal{G}(A_1^*) = V(\mathcal{G}(A_1))^\perp = \mathcal{G}(-C)^\perp = V(\mathcal{G}(-C)) = \mathcal{G}(A_1).$$

Therefore A_1 is selfadjoint.
Step 6: Conclusion. We let $A = A_1 - I$ with domain $\mathscr{D}(A) = \mathscr{D}(A_1)$. Then A is also selfadjoint and

$$\langle Au, v\rangle = \langle A_1 u, v\rangle - \langle u, v\rangle = (u, v)_{+1} - \langle u, v\rangle = q(u, v),$$

so that A is the operator associated with the form q. □

The following example shows that closedness is crucial.

2.5. Example. Let $H = L^2(\mathbb{R})$ and let q be the form

$$q(u, v) = u(0)\overline{v(0)}$$

with domain $Q(q) = C_c^\infty(\mathbb{R})$. This is a positive form, but there is no unbounded densely defined operator A on $L^2(\mathbb{R})$ such that

$$\langle Au, v\rangle = u(0)\overline{v(0)}.$$

In fact, suppose this were the case. Choose $u \in C_c^\infty(\mathbb{R})$ with $u(0) = 1$. Then we find a sequence (v_k) in $C_c^\infty(\mathbb{R})$ with $v_k(0) = 1$ and $v_k \to 0$ in $L^2(\mathbb{R})$. This furnishes the contradiction

$$1 = u(0)\overline{v_n(0)} \overset{?}{=} \langle Au, v_n\rangle \to 0.$$

2.b. **Weak derivatives and Sobolev spaces.**

2.6. Definition. Let Ω be an open subset of $\mathbb{R}^n$.

(a) Given $u, v \in L^1_{loc}(\Omega)$ and a multi-index α, we say that $v = \partial^\alpha u$, if

$$\int_\Omega u\partial^\alpha\varphi\,dx = (-1)^{|\alpha|}\int_\Omega v\varphi\,dx, \quad \varphi \in C_c^\infty(\Omega).$$

(b) For $k \in \mathbb{N}$ and $1 \le p \le \infty$, we define the Sobolev space

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p \text{ for all } |\alpha| \le k\}.$$

If $p = 2$, one often omits the superscript p. We define a norm on $W^{k,p}(\Omega)$ by

$$\|u\|_{k,p} = \sum_{|\alpha|\le k}\|\partial^\alpha u\|_p \text{ or, for } p = 2 : \|u\|_{k,2} = \Big(\sum_{|\alpha|\le k}\|\partial^\alpha u\|_2^2\Big)^{1/2}.$$

(c) $W_0^{k,p}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in the topology of $W^{k,p}(\Omega)$.

2.7. Theorem. *$W^{k,p}(\Omega)$ is a Banach space for $1 \le p \le \infty$ and a Hilbert space for $p = 2$. As a closed subspace, $W_0^{1,p}(\Omega)$ then also is a Hilbert space.*

Proof. It remains to check completeness. Suppose that (u_k) is a Cauchy sequence in $W^{k,p}$. Writing $u_k^\alpha = \partial^\alpha u_k$, this implies that (u_k^α) are Cauchy sequences in $L^p(\Omega)$. In view of the completeness of L^p, these sequences have limits u^α. We have to show that $\partial^\alpha u = u^\alpha$ (write $u^0 = u$) as this will imply that $u \in W^{k,p}$. To this end choose $\varphi \in C_c^\infty(\Omega)$. Then

$$\int_\Omega u\partial^\alpha\varphi\,dx = \lim_{k\to\infty}\int_\Omega u_k\partial^\alpha\varphi\,dx = \lim_{k\to\infty}(-1)^\alpha\int_\Omega u_k^\alpha\varphi\,dx = (-1)^\alpha\int_\Omega u^\alpha\varphi\,dx,$$

which is the assertion. □

2.8. Theorem. *For $1 \le p < \infty$ abd bounded Ω, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

Proof. See Evans, [2, §5.7, Theorem 1 and Remark on p.274]. □

2.9. Theorem. (Poincaré inequality, version 1) *Let Ω be a bounded domain in $\mathbb{R}^n$ (it is actually sufficient that Ω is bounded in one direction). Then there exists a constant $C = C(\Omega, p) \ge 0$ such that*

$$\|u\|_{L^p} \le C\|\nabla u\|_{L^p}, \quad u \in W_0^{1,p}(\Omega).$$

Proof. Let $\Omega \subseteq \{|x_1| \le R\}$. Integration by parts yields for $\varphi \in C_c^\infty(\Omega)$:

$$\|\varphi\|_{L^p}^p = \int_\Omega |\varphi(x)|^p 1\,dx = -\int_\Omega \partial_{x_1}(|\varphi(x)|^p)x_1\,dx$$

$$\le pR\int_\Omega |\partial_{x_1}\varphi(x)||\varphi(x)|^{p-1}\,dx \le pR\,\|\partial_{x_1}\varphi(x)\|_{L^p}\|\,|\varphi(x)|^{p-1}\,\|_{L^{p'}} \tag{1}$$

Since $p' = p/(p-1)$

$$\|\,|\varphi(x)|^{p-1}\,\|_{L^{p'}} = \left(\int_\Omega |\varphi(x)|^{(p-1)\frac{p}{p-1}}\right)^{(p-1)/p} = \|\varphi\|_{L^p}^{p-1}.$$

We thus obtain from (1)

$$\|\varphi\|_p \le pR\,\|\partial_{x_1}\varphi\|_{L^p} \le pR\|\nabla\varphi\|_{L^p}.$$

Since $C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ by assumption, the assertion follows. □

2.10. Theorem. (Poincaré inequality, version 2) *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary, $1 \le p \le \infty$. Then there exists a constant $C = C(\Omega, p) \ge 0$ such that*

$$\|u - u_\Omega\|_{L^p} \le C\|\nabla u\|_{L^p}, \quad u \in W^{1,p}(\Omega),$$

where

$$u_\Omega = \frac{1}{vol\,\Omega}\int_\Omega u\,dx$$

is the average of u over Ω.

Proof. See Evans, [2, §5.8, Theorem 1]. □

2.c. **The Dirichlet and Neumann problems.** The Dirichlet problem is to find, for given $f \in L^2(\Omega)$, a solution to the problem

(1) $$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

One also studies the Neumann problem, which requires to find, for given $f \in L^2(\Omega)$, a solution to the problem

(2) $$\Delta u = f \text{ in } \Omega, \quad \partial_\nu u = 0 \text{ on } \partial\Omega,$$

where ν is the exterior unit normal vector.

In these formulations, both problems require some regularity of the boundary. There is a way to circumvent this, namely the concept of weak solutions.

We call $u \in W_0^{1,2}(\Omega)$ a weak solution of the Dirchlet problem (1), provided that

(3) $$\int_\Omega \langle\nabla u, \nabla v\rangle_{\mathbb{C}^n}\,dx = -\int_\Omega f\overline{v}\,dx, \quad v \in W_0^{1,2}(\Omega).$$

Similarly, $u \in W^{1,2}(\Omega)$ is a weak solution of the Neumann problem (2), if

(4) $$\int_\Omega \langle\nabla u, \nabla v\rangle_{\mathbb{C}^n}\,dx = -\int_\Omega f\overline{v}\,dx, \quad v \in W^{1,2}(\Omega).$$

In the language of forms, we can define

$$\begin{aligned} q_D(u,v) &= \int_\Omega \langle\nabla u, \nabla v\rangle_{\mathbb{C}^n}\,dx, \quad Q(q_D) = W_0^{1,2}(\Omega) \\ q_N(u,v) &= \int_\Omega \langle\nabla u, \nabla v\rangle_{\mathbb{C}^n}\,dx, \quad Q(q_N) = W^{1,2}(\Omega). \end{aligned}$$

In fact, we have the same form with two different domains. The forms q_D and q_N are symmetric and positive. Since $(u,u)_{+1} = \|u\|_{W^{1,2}}$ and both $W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)$ are complete with respect to the associated norm, Theorem 2.4 applies. It furnishes two unbounded selfadjoint operators, namely $-\Delta_D$ and $-\Delta_N$, the Dirichlet and the Neumann Laplacian.

We can determine the domain of $-\Delta_D$ further:

2.11. Theorem. *$\mathscr{D}(-\Delta_D) = W_0^{1,2}(\Omega) \cap \{u \in L^2(\Omega) : \Delta u \in L^2(\Omega)\}$. This is $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, if $\partial\Omega$ is C^2-regular.*[1]

[1] $C^{1,1}$ suffices, see Grisvard [4, Theorem 2.2.2.3].

Proof. By the construction in Theorem 2.4 the domain of Δ_D consists of those $u \in Q(q_D) = W_0^{1,2}(\Omega)$, for which $Ju = jv$ for some $v \in H = L^2(\Omega)$. Explicitly this requires that

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle_{\mathbb{C}^n} dx + \int_\Omega u\overline{\varphi} dx = \int_\Omega v\overline{\varphi} dx, \quad \varphi \in H_{+1} = W_0^{1,2}(\Omega).$$

This shows that the divergence of ∇u exists and equals $u - v \in L^2$, hence $\Delta u = \operatorname{div} \nabla u \in L^2(\Omega)$. If the boundary is C^2, then one can show that $u \in W^{2,2}(\Omega)$, see Evans [2, §6.3, Theorem 4]. Conversely, for $\Delta u \in L^2(\Omega)$ and $u \in W_0^{1,2}(\Omega)$ the above identity holds, since it is true for $\varphi \in C_c^\infty(\Omega)$ which is dense in $W_0^{1,2}(\Omega)$ (apply Green's formula for a smoothly bounded domain in Ω containing $\operatorname{supp} \varphi$). □

2.12. Remark. Similarly as in Theorem 2.11, the domain of the Neumann Laplacian consists of all $u \in Q(q_N) = W^{1,2}(\Omega)$, for which $Ju = jv$ for some $v \in H = L^2(\Omega)$. Explicitly:

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle_{\mathbb{C}^n} dx + \int_\Omega u\overline{\varphi} dx = \int_\Omega v\overline{\varphi} dx, \quad \varphi \in H_{+1} = W^{1,2}(\Omega).$$

As before, we conclude that $\Delta u \in L^2(\Omega)$, but in this case, the above equality poses an additional restriction that is less obvious: If the boundary is smooth enough, Green's formula reveals that $\partial_\nu u = 0$.
If $\partial\Omega$ is of class $C^{1,1}$, then [4, Theorem 2.2.2.5] implies that $u \in W^{2,2}(\Omega)$.

Assumption. From now on assume that Ω is a bounded domain in $\mathbb{R}^n$.

2.13. Remark. Apart from the boundedness of Ω we shall assume in the following theorems that $\partial\Omega$ satisfies a certain regularity condition. It is sufficient that Ω has Lipschitz boundary or, slightly weaker, that it has the uniform cone property with finite cover, see [1, Section 4.4, p. 66]. That guarantees that elements of $W^{1,2}(\Omega)$ can be extended to elements of the standard Sobolev space $H^1(\mathbb{R}^n)$, see Definition 4.4; this is the so-called Calderón extension theorem, see [1, Theorem. 4.32]. As a consequence, the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

2.14. Theorem. *The Dirichlet Laplacian $-\Delta_D$ is a positive, invertible, selfadjoint operator on $L^2(\Omega)$. Its inverse is a compact, positive and selfadjoint operator on $L^2(\Omega)$, whose spectrum is a subset of $\overline{\mathbb{R}}_+$ with only accumulation point 0. Apart from 0, all spectral values are eigenvalues of finite multiplicity*
The spectrum of $-\Delta_D$ therefore consists of a sequence $0 = \lambda_1 < \lambda_2 < \ldots \to \infty$ of eigenvalues of finite multiplicity.

Proof. By construction, $-\Delta_D$ is positive and selfadjoint; it satisfies

$$\langle -\Delta_D u, v\rangle = q_D(u, v) = \int_\Omega \langle \nabla u, \overline{\nabla v}\rangle\, dx, \quad u, v \in W_0^{1,2}(\Omega).$$

We note that $-\Delta_D$ is injective: $-\Delta_D u = 0$ implies that $q(u, u) = 0$ and hence, by version 2 of Poincaré's inequality, that $u = 0$ a.e..
Moreover, $-\Delta_D$ has closed range: Suppose $u_k \in W_0^1(\Omega)$ and $(-\Delta_D u_k)$ is a Cauchy sequence in L^2. Then Poincaré's inequality implies that

$$\begin{aligned} \|u_k - u_l\|_{L^2}^2 &\le C\|\nabla(u_k - u_l)\|_{L^2}^2 = Cq_D(u_k - u_l, u_k - u_l) = C\langle -\Delta_D(u_k - u_l), u_k - u_l\rangle \\ &\le C\|\Delta_D(u_k - u_l)\|_{L^2}\|u_k - u_l\|_{L^2}. \end{aligned}$$

Hence (u_k) also is a Cauchy sequence in L^2. It has a limit $u \in L^2$. Since $-\Delta_D$ is closed, $u \in \mathscr{D}(-\Delta_D)$ and $\lim \Delta_D u_k \to \Delta_D u$.
From this follows surjectivity: Suppose, $-\Delta_D$ were not surjective. As the range is closed, we could find $0 \ne v \in L^2$ such that $\langle -\Delta_D u, v\rangle = 0$ for all $u \in \mathscr{D}(-\Delta_D)$. This implies that $v \in$

$\mathscr{D}(-\Delta_D^*) = \mathscr{D}(-\Delta_D)$. Choosing $u := v$ we obtain from Poincaré's inequality the contradiction $0 = \langle -\Delta_D v, v\rangle = q(v,v) \geq \|v\|^2 > 0$.

So we know that $-\Delta_D : \mathscr{D}(-\Delta_D) \subseteq L^2 \to L^2$ is positive, selfadjoint and invertible. Its inverse is a bounded, positive and selfadjoint operator on L^2.

It is even compact, since its range is $\mathscr{D}(-\Delta_D) \subseteq W_0^{1,2}(\Omega)$, and, by Theorem 2.8, $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. The spectrum of a positive, selfadjoint compact operator is a subset $\{\mu_1 > \mu_2 > \ldots\} \cup \{0\}$ of $\overline{\mathbb{R}}_+$ with only possible accumulation point zero. Apart from zero, all spectral values are eigenvalues of finite multiplicity.

The spectrum of $-\Delta_D$ then consists of the values μ_j^{-1}, $j = 1, 2, \ldots$ and therefore has the stated properties. (Actually, we see just as above, that $\lambda_j I - \Delta_D$ is not invertible if and only if it is not injective; so all spectral values are eigenvalues and thus coincide with the μ_j^{-1}). □

2.15. Theorem. *The Neumann Laplacian $-\Delta_N$ is a positive selfadjoint operator on $L^2(\Omega)$. Its spectrum consists of a sequence $0 = \lambda_1 < \lambda_2 < \ldots \to \infty$ of eigenvalues of finite multiplicity. Note that $-\Delta_N$ is not invertible; it has a one-dimensional kernel.*

Proof. Consider the scalar product $q_{+1}(u,v) = \langle u, v\rangle + q_N(u,v)$ on $W^{1,2} \times W^{1,2}$, which is associated with the operator $-\Delta_N + I$. A similar, but simpler argument than that for $-\Delta_D$ then shows that $I - \Delta_N$ is invertible and selfadjoint. Its inverse is compact, since it maps into $W^{1,2}(\Omega)$ which embeds compactly into $L^2(\Omega)$. Hence the spectrum of $I - \Delta_N$ consists of eigenvalues of finite multiplicity; they tend to $+\infty$. In view of the fact that $\langle (I - \Delta_N)u, u\rangle = q_{+1}(u,u) \geq \|u\|^2$, the eigenvalues of $I - \Delta_N$ must be ≥ 1.

It is clear that $q_N(c,c) = 0$ for constant functions c, so that 1 is an eigenvalue. In order to see that it is a simple eigenvalue note that $q_{+1}(u,u) = \|u\|^2$ implies that $q_N(u,u) = 0$; hence u is locally constant by version 2 of Poincaré's inequality, applied to any ball in Ω, thus constant, since Ω is connected. Subtraction of the identity operator then implies the assertion. □