1. Background on Functional Analysis

We start by recalling a few basics from any course on Functional Analysis. Suitable sources for this material are the books of D. Werner [9] or W. Rudin [7] (more advanced) or my course notes on Functional Analysis.

Let H, H_1, H_2 be Hilbert spaces. We write generically $\langle\cdot,\cdot\rangle$ for the scalar product.

All statements in this section that do not directly involve the Hilbert space structure actually also hold in the Banach space setting.

1.a. **Bounded and compact operators, spectra.**

1.1. Definition. We call a linear operator $A : H_1 \to H_2$ bounded and write $A \in \mathcal{L}(H_1, H_2)$, if there exists a constant $C \geq 0$ such that

$$\|Ax\| \leq C\|x\|, \ x \in H_1.$$

1.2. Lemma. *For linear operators, boundedness is equivalent to continuity.*

1.3. Theorem. **(Riesz representation theorem)**. *For every continuous linear functional $F : H \to \mathbb{C}$ there exists precisely one element $v_F \in H$ such that*

$$F(u) = \langle u, v_F\rangle.$$

This provides an isometric conjugate-linear map from $\mathcal{L}(H, \mathbb{C})$ to H.

Equivalently, we find for every continuous conjugate-linear map $G : H \to \mathbb{C}$ precisely one element v_G such that

$$G(u) = \langle v_G, u\rangle.$$

This furnishes an isometric isomorphism from the space of conjugate-linear functionals on H to H.

1.4. Theorem. **(Closed graph theorem)** *Let $A : H_1 \to H_2$ be a linear operator. Then A is continuous if and only if the graph of A, i.e. the set $\{(x, Ax) : x \in H_1\}$, is closed in $H_1 \times H_2$.*

In other words: A is bounded, if and only if for every sequence (x_k) in H_1 with $x_k \to 0$ in H_1 and $Ax_k \to y$ in H_2, we have $y = 0$.

1.5. Theorem. **(Hellinger-Toeplitz)** *Let $A : H \to H$ be a linear operator satisfying*

$$\langle Ax, y\rangle = \langle x, Ay\rangle, x, y \in H.$$

Then A is continuous.

Proof. Assume that $x_k \to 0$ and $Ax_k \to z$. Then we have for all $y \in H$:

$$\langle z, y\rangle = \lim\langle Ax_k, y\rangle = \lim\langle x_k, Ay\rangle = 0.$$

Hence $z = 0$ and the closed graph theorem implies the assertion. □

1.6. Definition. We call $K \in \mathcal{L}(H_1, H_2)$ compact, if it maps bounded sets into compact sets. Write $K \in \mathcal{K}(H_1, H_2)$.

We say that H_1 is compactly embedded in H_2, if the identity operator $I : H_1 \to H_2$ is compact.

1.7. Remark.

(a) Since closed subsets of compact subsets of Banach spaces are compact, we can rephrase this in several ways:
- If $M \subset H_1$ is bounded, then the closure $\overline{K(H_1)} \subset H_2$ is compact.
- The closure of the image of the unit ball $\overline{K(B(0,1)}$ is compact.

- If $(x_k)_k$ is a bounded sequence in H_1, then $(Kx_k)_k$ has a convergent subsequence.

(b) $\mathcal{K}(H_1, H_2)$ is a closed subspace of $\mathcal{L}(H_1, H_2)$.

(c) The closed unit ball in a Banach space is compact if and only if its dimension is finite. As a consequence, every operator of finite rank is compact.

(d) If K is compact, A and B are bounded, then AK and KB are compact.

1.8. Definition. The spectrum $\boldsymbol{\sigma}(A)$ of an operator $A \in \mathcal{L}(H)$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A : E \to E$ is not bijective. The complement $\mathbb{C} \setminus \boldsymbol{\sigma}(A)$ is called the resolvent of A.

1.9. Theorem.

(a) *The spectrum of a bounded linear operator $A \in \mathcal{L}(H)$ is a nonempty compact subset of $\mathbb{C}$. Conversely, every nonempty compact subset of $\mathbb{C}$ arises as the spectrum of a bounded linear operator.*

(b) *If $A \in \mathcal{L}(H_1, H_2)$ is bijective, then $A^{-1} \in \mathcal{L}(H_2, H_1)$ (automatic continuity of the inverse).*

1.10. Theorem. *Let $K \in \mathcal{K}(H)$. Then $\boldsymbol{\sigma}(K)$ is a bounded, discrete subset of $\mathbb{C}$ with only possible accumulation point 0. If $\dim H = \infty$, then $0 \in \boldsymbol{\sigma}(K)$. Except for 0, all points in $\boldsymbol{\sigma}(K)$ are eigenvalues of finite multiplicity.*

1.11. Theorem. *Given a linear operator $A \in \mathcal{L}(H)$ there exists a unique operator $A^* \in \mathcal{L}(H)$ such that*

$$\langle Ax, y\rangle = \langle x, A^*y\rangle, \quad x, y \in H.$$

Furthermore $\|A^*\| = \|A\|$.
We call A selfadjoint if $A^* = A$. In this case we have

$$\|A\| = \max\{|\lambda| : \lambda \in \boldsymbol{\sigma}(A)\}.$$

1.b. **Unbounded operators.**

1.12. Definition. An unbounded linear operator is a linear map $T : \mathscr{D}(T) \subseteq H \to H$. The subspace $\mathscr{D}(T)$ is called the domain of T. When speaking of unbounded operators we will always assume linearity.
The operator $T : \mathscr{D}(T) \to H$ is *closed*, if the graph of T, i.e. the set

$$\mathcal{G}(T) = \{(x, Tx) : x \in \mathscr{D}(T)\}$$

is a closed subset of $H \times H$.

1.13. Lemma. *We can endow $\mathscr{D}(T)$ with the norm $\|x\|_T = \|x\| + \|Tx\}$. If T is closed, then $(\mathscr{D}(T), \|\cdot\|_T)$ is a Banach space, and*

$$T : (\mathscr{D}(T), \|\cdot\|_T) \to T$$

is a bounded linear operator. In this sense, closedness is a substitute for boundedness.

Proof. Clearly, $\|\cdot\|_T$ is a norm. If (x_k) is a Cauchy sequence in $(\mathscr{D}(T), \|\cdot\|_T)$, then (x_k) is a Cauchy sequence in H thus has a limit x, and (Tx_k) is a Cauchy sequence in H and has a limit y. Thus (x, y) is the limit of a sequence in the graph of T. Since the graph is closed, $(x, y) \in \mathcal{G}(T)$. We conclude that $x_k \to x \in (\mathscr{D}(T), \|\cdot\|_T)$, so that this space is complete. Now the closed graph theorem implies the boundedness of T in this setting.

1.14. Definition. The operator $T : \mathscr{D}(T) \subset H \to H$ is *invertible*, if there exists an operator $Q \in \mathcal{L}(H)$ with range $\mathscr{D}(T)$ such that $TQ = I_H$ and $QT = I_{\mathscr{D}(T)}$. The *spectrum* of T is the set of all $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is not invertible. The complement of the spectrum is called the *resolvent (set)*.

1.15. Remark. (a) The spectrum always is a closed subset of $\mathbb{C}$, and every closed subset arises as the spectrum of an unbounded operator.
(b) If T is not closed, then $\boldsymbol{\sigma}(T) = \mathbb{C}$.

1.16. Definition. We call $T : \mathscr{D}(T) \to H$ closable, if there exists a closed extension of T, i.e. an operator $\widehat{T}$ with domain $\mathscr{D}(\widehat{T}) \supseteq \mathscr{D}(T)$ such that $\widehat{T}_{|\mathscr{D}(T)} = T$. In this case we call the smallest closed extension the *closure* of T and denote it by $\overline{T}$.

1.17. Lemma. *The following are equivalent for an unbounded operator T with domain $\mathscr{D}(T)$:*
(a) *T is closable.*
(b) *The closure $\overline{\mathcal{G}(T)}$ of the graph is the graph of an operator $\overline{T}$*
(c) *If (x_k) is a sequence in $\mathscr{D}(T)$ with $x_k \to 0$ and $Tx_k \to y$, then $y = 0$.*
In this case, the closure $\overline{T}$ is defined by

(1) $\mathscr{D}(\overline{T}) = \{x \in H : \exists (x_k) \subset \mathscr{D}(T) \text{ such that } x_k \to x \text{ and } Tx_k \to y \text{for some } y \text{ in } H\}.$

Proof. (a) $\Rightarrow$ (b): If $\overline{T}$ is the closure of T, then $\overline{\mathcal{G}(T)}$ is the graph of $\overline{T}$.
(b) $\Rightarrow$ (c): Follows from the fact that $\overline{T}0 = 0$.
(c) $\Rightarrow$ (a): Define $\mathscr{D}(\overline{T})$ by (1) and let $\overline{T}x = y$ whenever (x, y) is the limit of a sequence (x_k, Tx_k) with $x_k \in \mathscr{D}(T)$.
This makes sense: If $(x_k, Tx_k) \to (x, y)$ and $(\tilde{x}_k, T\tilde{x}_k) \to (x, \tilde{y})$, then $x_k - \tilde{x}_k \to 0$ and $T(x_k - \tilde{x}_k) \to y - \tilde{y}$, so that, by assumption, $y = \tilde{y}$. Moreover, it defines a linear operator. By construction, its graph is closed. □

1.18. Definition. Let T be closed. We call a subset $\mathscr{C}$ of $\mathscr{D}(T)$ a *core* for T, if $\overline{T_{|\mathscr{C}}} = T$.

1.19. Definition. Let $T : \mathscr{D}(T) \to H$ be densely defined.
(a) T is called *symmetric*, if

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \quad x, y \in \mathscr{D}(T).$$

(b) The *adjoint* T^* of an unbounded operator T is defined on the set

$$\mathscr{D}(T^*) = \{x \in H : \exists z \in H \text{ s.t. } \langle x, Ty\rangle = \langle z, y\rangle, y \in \mathscr{D}(T)\}.$$

In this notation one lets $T^*x = z$.
(c) T is called *selfadjoint* if $\mathscr{D}(T^*) = \mathscr{D}(T)$ and $T = T^*$ on $\mathscr{D}(T)$.
(d) T is called *essentially selfadjoint* if its closure is selfadjoint.

1.20. Remark. (a) We need that T is densely defined in order to have a unique z in 1.19(b).
(b) 'Symmetric' and 'selfadjoint' are different notions. If T is symmetric, then $\mathscr{D}(T) \subseteq \mathscr{D}(T^*)$, so that T^* is an extension of T. In general, however, a symmetric operator might not have a selfadjoint extension.

1.21. Theorem. *Let T be densely defined. Then*
(a) *T^* is closed.*
(b) *T is closable, if and only if $\mathscr{D}(T^*)$ is dense in H. In this case, $\overline{T} = T^{**}$.*
(c) *If T is closable, then $(\overline{T})^* = T^*$.*

Proof. Define a scalar product on $H \times H$ by

$$((x, y), (u, v)) = \langle x, u\rangle + \langle y, v\rangle$$

and a map

$$V : H \times H \to H \times H \text{ by } V(x, y) = (-y, x).$$

Then $V^2 = -I$ and for every subspace U of $H \times H$

(1) $$V(U^\perp) = V(U)^\perp.$$

(a) Note that

$$\begin{aligned} (u,v) \in V(\mathcal{G}(T))^\perp &\Leftrightarrow \langle u, -Tx\rangle + \langle v, x\rangle = 0 \ \forall x \in \mathscr{D}(T) \\ &\Leftrightarrow \langle u, Tx\rangle = \langle v, x\rangle \ \forall x \in \mathscr{D}(T) \\ &\Leftrightarrow (u,v) \in \mathcal{G}(T^*) \end{aligned}$$

Hence

(2) $$\mathcal{G}(T^*) = V(\mathcal{G}(T))^\perp$$

which is a closed subset of $H \times H$.

(b) We have $\overline{M} = (M^\perp)^\perp$ for every subspace M of a Hilbert space. Hence we always have

(3) $$\overline{\mathcal{G}(T)} = (\mathcal{G}(T)^\perp)^\perp \overset{V^2=-I}{=} (V^2(\mathcal{G}(T)^\perp))^\perp \overset{(1)}{=} (V(V(\mathcal{G}(T)))^\perp)^\perp \overset{(3)}{=} V(\mathcal{G}(T^*))^\perp.$$

So, if T^* is densely defined, we can apply Equation (2) of (a) and find that $\overline{\mathcal{G}(T)} = \mathcal{G}(T^{**})$. Conversely, suppose that T is closable, but $\mathscr{D}(T^*)$ is not dense in H. Then we find $0 \neq y \in \mathscr{D}(T^*)^\perp$. This implies that $(y,0) \in \mathcal{G}(T^*)^\perp$. Hence

$$(0,y) \in V(\mathcal{G}(T^*)^\perp) \overset{(1)}{=} V(\mathcal{G}(T^*))^\perp \overset{(3)}{=} \overline{\mathcal{G}(T)} \overset{(1.17)(b)}{=} \mathcal{G}(\overline{T}).$$

This is a contradiction, since a linear operator maps zero to zero and $y \neq 0$.

(c) If T is closable, then $T^* \overset{(a)}{=} \overline{T^*} \overset{(b)}{=} T^{***} = \overline{T}^*$. $\square$