9. Proof of the K-theoretic Index Theorem

9.1. The situation. Let M be a compact manifold. We shall assume that M is endowed with a Riemannian metric that allows us to identify TM and T^*M.

Furthermore, let E and F be complex vector bundles over M and

$$P : C^\infty(M, E) \to C^\infty(M, F)$$

an elliptic pseudodifferential operator. For simplicity, let us fix the order of P to be zero.

Denote by $\pi : T^*M \to M$ and $\pi_0 : T^*M \setminus 0 \to M$ the base point projections. By definition, the principal symbol

$$\sigma(P) : \pi_0^* E \to \pi_0^* F$$

then is an isomorphism. By possibly using an excision function ρ as in 8.4, we have

$$[\sigma(P)] = [(\pi^*E, \pi^*F, \sigma(P))] \in K_c(T^*M) \cong K_c(TM).$$

9.2. Outline. According to Theorem 5.35. P is a Fredholm operator with an index $\operatorname{ind} P \in \mathbb{Z}$.

Below, we will see how we can moreover associate a so-called topological index $\operatorname{ind}_t[\sigma(P)] \in \mathbb{Z}$ to the element $[\sigma(P)]$ in $K_c(TM)$. The index theorem states that these two numbers are the same.

In fact, one changes the perspective slightly: Given the element $[\sigma(P)] \in K_c(TM)$ we can pick any zero order pseudodifferential operator $\tilde{P} : C^\infty(M, E) \to C^\infty(M, F)$ with principal symbol $\sigma(\tilde{P}) = \sigma(P)$. From Theorem 5.40 we know that

$$\operatorname{ind} \tilde{P} = \operatorname{ind} P,$$

so that it makes sense to associate to $[\sigma(P)]$ the analytic index

$$\operatorname{ind}_a[\sigma(P)] = \operatorname{ind} \tilde{P}.$$

Then the task changes to showing that

$$\operatorname{ind}_a[\sigma(P)] = \operatorname{ind}_t[\sigma(P)].$$

In fact, as we shall see in Lemma 9.9, below, every element of $K_c(T^*M)$ is of the form $[(\pi^*E, \pi^*F, a)]$ for suitable complex vector bundles E, F over M and a bundle morphism $a : \pi^*E \to \pi^*F$ which is an isomorphism outside a compact set and homogeneous of degree zero in the fiber. So we will actually show that

$$\operatorname{ind}_t = \operatorname{ind}_a : K_c(T^*M) \to \mathbb{Z}.$$

9.3. Theorem: Another picture for K_c. We have the following equivalent representation for $K_c(X)$ whenever X is a locally compact Hausdorff space. We choose an additional point 'P' and denote by X^+ the one-point compactification of X.[16] This is a compact space. We can therefore consider

[16] As a set, X^+ is $X\dot{\cup}\{P\}$. The open sets are given by the open sets in X and all sets of the form $\{P\}\dot{\bigcup}(X \setminus K)$, where K subset X is compact.

the usual K-theory on X^+. Moreover, the embedding $\iota : P \hookrightarrow X^+$ induces the evaluation map $\iota^* : K(X^+) \to K(\{P\})$. We let

$$K_c(X) = \ker(\iota^* : K(X^+) \to K(\{P\})) \subset K(X^+).$$

Proof. Every vector bundle in the newly introduced definition is of the form $[(E, F)]$, where E and F are complex vector bundles over X^+. The fact that the class is in the kernel of ι^* implies that E_P and F_P are isomorphic. By continuity, there exists a neighborhood U of P such that also E_U and F_U are isomorphic via some vector bundle isomorphism a. Since $X^+ \setminus U$ is a compact subset of X, we obtain a triple (E, F, a) in the previous sense.

Conversely, let (E, F, a) be a representative for a class in the previous definition of $K_c(X)$. XXX

9.4. Lemma. *Let Y be a locally compact Hausdorff space and X an open subset of Y. Then we have a natural extension map $F : K_c(X) \to K_c(Y)$.*

Proof. We can identify X^+ with the quotient space $Y^+/(Y^+ \setminus X)$. The natural map $\phi : Y^+ \to Y^+/(Y^+ \setminus X) = X^+$ then induces the map $\Phi : K(X^+) \to K(Y^+)$. In view of the fact that the map preserves the additional point P, the image of a class in $K_c(X)$ is a class in $K_c(Y)$ so that Φ restricts to F.

9.5. Lemma. *With every proper embedding*[17] *$f : X \hookrightarrow Y$ we can associate a map*

$$f_! : K_c(TX) \to K_c(TY).$$

Proof. The embedding $f : X \hookrightarrow Y$ induces a proper embedding $f_* : TX \hookrightarrow TY$ by associating to a path $\gamma : [-1, 1] \to X$ its image $f \circ \gamma$.

Let N_X be the normal bundle to $f(X)$ in Y[18] Then the normal bundle N_{TX} to $f_*(TX)$ in TY is the pull-back of $N_X \oplus N_X$ to $T(f(X))$, where the first summand is in the direction of the manifold, the second in the direction of the fiber. Note that $T(T_xY) \cong T_xY$ naturally.

We observe that $N_{TX} \cong N_X \oplus N_X$ has a complex structure, given by

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Of course, $K_c(TX)$ and $K_c(f_*(TX))$ are naturally isomorphic. We can therefore consider N_{TX} as a complex vector bundle over TX and apply the Thom isomorphism

$$i_! : K_c(TX) \to K_c(N_{TX}).$$

As a normal bundle, N_{TX} can be identified with an open neighborhood of $f(TX)$ in TY. Lemma 9.4 thus shows that we have a natural extension map $F : K_c(N_{TX}) \to K_c(TY)$. Their composition yields the desired map

$$f_! : K_c(TX) \to K_c(TY).$$

□

[17] A map is proper, if the preimage of every compact set is compact

[18] i.e. N_x, $x \in X$ is the space of all $u \in T_xY$ such that $\langle u, v \rangle = 0$ for all $v \in T_xX$.

9.6. Whitney embedding theorem. Any manifold M can be embedded into $\mathbb{R}^K$ for suitably large K. One can show that $K = 2\dim M$ suffices.

Proof. We can cover M by compact sets K_j, $j = 1, \ldots, J$, contained in coordinate neighborhoods M_j with coordinate maps $\kappa_j : M_j \to U_j \subseteq \mathbb{R}^n$. Choose $\varphi_j \in C_c^\infty(M_j)$ with $\varphi_j \equiv 1$ on K_j, and define

$$F : X \to \mathbb{R}^{(n+1)J} \text{ by } F(x) = (\varphi_1(x), \varphi_1(x)\kappa_1(x), \ldots, \varphi_J(x), \varphi_J(x)\kappa_J(x)).$$

This map is injective: If $x \in K_j$ and $F(y) = F(x)$, then $\varphi_j(y) = \varphi(x) = 1$. Hence $x \in M_j$ and $\kappa_j(x) = \kappa_j(y)$ and thus $x = y$. Similarly, $F'(x)$ is injective. Therefore M is a submanifold of $R^{(n+1)J}$. □

9.7. The topological index. Identify TM and T^*M, consider $[\sigma(P)]$ as an element of $K_c(TM)$, choose an embedding f of M into $\mathbb{R}^K$ for some large K and consider the induced map

$$f_! : K_c(TM) \to K_c(T\mathbb{R}^K).$$

In view of the fact that $T\mathbb{R}^K \cong \mathbb{C}^N$ is a complex vector bundle over the point 0, we have the inverse of the Thom isomorphism

$$(i_!)^{-1} : K_c(T\mathbb{R}^K) \to K(\{0\}).$$

Since the K-theory of a single point is naturally isomorphic to $\mathbb{Z}$, we have, in a natural way, the topological index map

$$\text{ind}_t := (i_!)^{-1} f_! : K_c(TM) \to \mathbb{Z}.$$

9.8. Remark. Landweber [15, p.17] (adapted from Atiyah-Singer [3, p.498]) argues as follows for the independence of the embedding: Suppose that we have two embeddings $i : M \to \mathbb{R}^m$ and $i' : M \to \mathbb{R}^n$. Considering the diagonal embedding $i \oplus i' : M \to \mathbb{R}^{m+n}$, we see that $i \oplus i'$ is isotopic to the embeddings $i \oplus 0$ and $0 \oplus i'$. Letting $\beta_n \in K(\mathbb{C}^n)$ denote the Thom class of $\mathbb{C}^n = T\mathbb{R}^n$, we note that $(i \oplus 0)_!(x) = i_!(x) \cdot \beta_n$ and $(0 \oplus i')_!(x) = \beta_m \cdot i'_!(x)$. We then see by the transitivity of the Thom isomorphism that $i \oplus 0$ and i determine the same topological index, as do $0 \oplus i'$ and i'. It follows that the topological index does not depend on our choice of embedding.

In order to define the analytic index on $K_c(TM)$ we need the following result:

9.9. Lemma. *Let M be compact and V a smooth real vector bundle over M, e.g. $V = TM$. Then every element of $K_c(V)$ has a representative the form $(\pi^*F^0, \pi^*F^1, \sigma)$, with complex vector bundles F^0, F^1 over M, the projection map $\pi : V \to M$ and a vector bundle homomorphism σ which outside a compact set is an isomorphism and homogeneous of degree 0 (and hence can be continued zero-homogeneously to $V \setminus 0$).*

Proof. Given a class in $K_c(V)$ choose a representative (E^0, E^1) in $K(V^+)$ with $E^0_P \cong E^1_P$. There is a neighborhood U of the additional point over which E^0 and E^1 are isomorphic. We can endow V with a Riemannian metric and assume this neighborhood to be the complement of the ball bundle $B_r(V) = \{(x, v) \in V : x \in M, \|v\| < r\}$ for some $r > 0$. The classes of E^0 and E^1 only depend on the restriction to $B_r(V)$. Inside $B_r(V)$ we let[19] $F^j = \pi^* E^j_M$ for $j = 0, 1$, and claim that the bundles E^j and F^j are isomorphic. In fact, following [1, Lemma 1.4.5], define the map

$$H : B_r(V) \times [0,1] \to B_r(V), \quad (x,v) \mapsto (x, tv)$$

and the bundles $\mathscr{E}^j = H^* E^j$ over $B_r(V) \times [0,1]$. Then

$$\mathscr{E}^j_{|B_r(V)\times\{0\}} = \pi^* E^j_M = F^j \text{ and } \mathscr{E}^j_{|B_r(V)\times\{1\}} = E^j.$$

This shows that there are isomorphisms $\gamma_j : E^j \stackrel{\cong}{\to} F^j$ over $B_r(V)$. Denote by $\alpha : E^0 \to E^1$ the isomorphism between the bundles on the boundary $S_r(V) = \{(x,v) : \|v\| = r\}$ of $B_r(V)$. Then we obtain an isomorphism $\sigma = \gamma_1 \alpha \gamma_0^{-1}$ between F^0 and F^1 on $S_r(V)$. As these are pull-backs from the base, we can extend σ homogeneously to $V \setminus 0$. Then the class of $(\pi^* F^0, \pi^* F^1, \sigma)$ coincides with that of (E^0, E^1, α), since the class only depends on the restriction to $B_r(V)$. □

9.10. Definition. Given a class in $K_c(TM)$, Lemma 9.9 allows us to find a representative of the form $(\pi^* F^0, \pi^* F^1, \sigma)$, where σ is 0-homogeneous. As pointed out already in 9.2, we can associate to this representative a zero order pseudodifferential operator $P : C^\infty(M; F^0) \to C^\infty(M, F^1)$ with symbol σ and to this pseudodifferential operator its index, which does not depend on the choices. This is the analytic index map.

9.11. Remark. It remains to check that this is independent of the choice of the representative. See Lawson-Michelsohn, [16, p. 247] for an argument.

9.12. Lemma. *The topological index can be seen as a map, which associates to a compact manifold M a map $\mathrm{ind}_t = \mathrm{ind}_t^M : K_c(TM) \to \mathbb{Z}$. It satisfies the following two properties:*

(A) *If M is a point (and hence TM is a point), then $ind_t : K_c(TM) \cong K_c(\{0\}) = K(\{0\}) \to \mathbb{Z}$ is the identity map.*

(B) *If $f : M \hookrightarrow M'$ is an embedding of compact manifolds with the associated map $f_! : K_c(TM) \to K_c(TM')$ constructed in 9.5, then*

$$\mathrm{ind}_t^{M'}(f_! u) = \mathrm{ind}_t^M(u), \quad u \in K_c(TM).$$

Proof. (A) holds by definition.

(B) follows from the fact that the Thom isomorphism satisfies $(ii')_! = i_! i'_!$. □

9.13. Observation. In order to establish the K-theoretic version of the index theorem, it suffices to show that the analytic index also satisfies (A) and (B) in Lemma 9.12.

Proof. Indeed, suppose this is true. Consider the embeddings $i : M \hookrightarrow \mathbb{R}^K$ for some large K and $j : P \to \mathbb{R}^K$, where P is a single point.

We note that $\mathbb{S}^K = (\mathbb{R}^K)^+$ is the one-point compactification. Denote by $i^+ : M \hookrightarrow \mathbb{S}^K$ and $j^+ : P \hookrightarrow \mathbb{S}^K$ the associated embeddings. Consider the

[19] Here we view M as the zero section of V

diagram

$$
\begin{array}{ccccc}
 & & K_c(T\mathbb{R}^K) & & \\
 & \overset{i_!}{\nearrow} & \big\downarrow F & \overset{j_!}{\nwarrow} & \\
K_c(TM) & \xrightarrow{i_!^+} & K_c(T\mathbb{S}^K) & \xleftarrow{j_!^+} & K_c(TP) \\
 & \underset{\mathrm{ind}_a^M}{\searrow} & \big\downarrow \mathrm{ind}_a^{\mathbb{S}^K} & \underset{\mathrm{ind}_a^P}{\swarrow} & \\
 & & \mathbb{Z}, & &
\end{array}
$$

where F is the extension map from Lemma 9.4. The top two triangles then commute by definition of $i_!$ and $j_!$. The bottom two triangles commute by (B). According to (A), ind_a^P is the identity. Since $j_!$ is an isomorphism, and $\mathrm{ind}_t : K_c(TM) \to \mathbb{Z}$ is given by $(j_!)^{-1} i_!$, the commutativity of the diagram shows that it coincides with ind_a. □

9.14. The analytic index satisfies (A) in Lemma 9.7. In fact, a pseudodifferential operator acting on sections of complex vector bundles over a manifold, which consists of a single point is just a linear map $P : \mathbb{C}^{n_1} \to \mathbb{C}^{n_2}$. As we saw in the beginning of Section 2, the index then is $n_1 - n_2$ which coincides with the K-theoretic map.

9.15. Proving property (B). We recall that the map $K_c(TM) \to K_c(TM')$ associated to a proper embedding $f : M \to M'$ is the composition

$$K_c(TM) \overset{i_!}{\to} K_c(N_{TM}) \overset{F}{\to} K_c(TM'),$$

where $i_!$ is given by the Thom isomorphism and F is the extension map of Lemma 9.4 which we apply after considering the normal bundle N_{TM} to $f_*(TM)$ in TM' as an open neighborhood of $f_*(TM)$.

Now we observe that the normal bundle N_{TM} can also be considered as TN, where N is the normal bundle to M, so that we obtain the following diagram

$$
(1) \qquad
\begin{array}{ccccc}
K_c(TM) & \xrightarrow{i_!} & K_c(TN) & \xrightarrow{F} & K_c(TM') \\
 & \underset{\mathrm{ind}_a^M}{\searrow} & \big\downarrow \mathrm{ind}_a^N & \underset{\mathrm{ind}_a^{M'}}{\swarrow} & \\
 & & \mathbb{Z}. & &
\end{array}
$$

Property (B) will be shown once we establish the commutativity of both triangles. The following lemma addresses the right triangle.

9.16. Lemma. *The right triangle in 9.15*(1) *commutes.*

Proof. (Sketch). In a first step one shows that the statement of Lemma 9.9 extends to the case where the base U (in our case, $U = N$) is non-compact as follows: Every class in $K_c(TU)$ has a representative $(\pi^* E^0, \pi^{*1}, \sigma)$ where E^0 and E^1 are bundles over U that are trivial outside a compact subset of U and where σ is homogeneous of degree zero outside a compact subset of T^*U, see [16, Lemma 13.3].

In particular, outside a compact subset L of U, one has trivializations

$$\tau_{E^0} : E^0_{|U \setminus L} \xrightarrow{\cong} (U \setminus L) \times \mathbb{C}^m \text{ and } \tau_{E^1} : E^1_{|U \setminus L} \xrightarrow{\cong} (U \setminus L) \times \mathbb{C}^m$$

with respect to which $\sigma_{x,\xi} = (\tau^1_E)^{-1}_x \circ (\tau^0_E)_x$, independent of ξ for $(x,\xi) \in T^*(U \setminus L)$. This means that, over $T^*(U \setminus L)$, the bundle map σ comes from a bundle map σ_0 over the base. Moreover, with respect to the above trivializations, σ_0 becomes the identity mapping over $U \setminus L$. We then find a zero order *differential* operator $P : C^\infty(U, E^0) \to C^\infty(U, E^1)$ which is the identity outside a compact set (the operator can be chosen differential since σ comes from the map σ_0 on the base, i.e. is just a morphism.

Applying this to the case where $U = N \overset{F}{\hookrightarrow} M'$, and $(\pi^* E^0, \pi^* E^1, \sigma) \in K_c(N)$, we extend E^0 and E^1 by the trivial bundle $\mathbb{C}^m$ to bundles $\underline{E}^0$ and $\underline{E}^1$ over M' and extend σ as the identity. This gives us a class $(\pi^*\underline{E}^0, \pi^*\underline{E}^1, \underline{\sigma}) = F_!(\pi^*E^0, \pi^*E^1, \sigma)$ in $K_c(T^*M')$. We can also extend the above operator P by the identity. This furnishes an elliptic operator $F_!P$ on M'.

Now assume $C^\infty(N, E^0) \ni u \in \ker P$. The fact that P is the identity outside the compact set L implies that $\operatorname{supp} u \subset L$. Hence u extends, by zero, to a function in $C^\infty(M', \underline{E}^0)$ with $P'u = 0$. Hence $\dim \ker P \le \dim \ker F_!P$. Conversely, if $C^\infty(M', \underline{E}^0) \ni v$ and $F_!Pv = 0$, then, by the same argument as before, $\operatorname{supp} v \subset N$, and thus $v_{|N} \in \ker P$. One concludes that $\dim \ker P = \dim \ker F_!P$. A corresponding argument applies to the kernels of the adjoints, so that the analytic indices of P and $F_!P$ agree. □

9.17. Remark. To show that the left triangle in 9.15(1) also commutes requires a much lengthier argument.