8. Basics of K-theory

8.1. The ring $K(M)$.

(i) A *virtual bundle* over a (not necessarily compact) manifold M is a pair $V = (E^0, E^1)$ of vector bundles over M.

(ii) V is said to be *trivial*, if E^0 and E^1 are isomorphic.

(iii) Two virtual bundles $V_1 = (E_1^0, E_1^1)$ and $V_2 = (E_2^0, E_2^1)$ are *isomorphic*, if $E_1^0 \cong E_2^0$ and $E_1^1 \cong E_2^1$ are isomorphic bundles.

(iv) On the set of these pairs we define the operations $\oplus$ and $\otimes$ of *direct sum* and *tensor product* for virtual bundles V_1 and V_2 as above by

$$\begin{aligned} V_1 \oplus V_2 &= (E_1^0 \oplus E_2^0, E_1^1 \oplus E_2^1) \\ V_1 \otimes V_2 &= ((E_1^0 \otimes E_2^0) \oplus (E_1^1 \otimes E_2^1), (E_1^0 \otimes E_2^1) \oplus (E_1^1 \otimes E_2^0)). \end{aligned}$$

(v) We call V_1 and V_2 *stably isomorphic* and write $V_1 \sim V_2$, if there exist trivial virtual bundles W_1 and W_2 such that

$$V_1 \oplus W_1 \cong V_2 \oplus W_2.$$

(vi) The direct sum and the tensor product define an addition and a multiplication on the equivalence classes of virtual bundles with respect to $\sim$. There exists a unit element, given by the class of $(0,0)$. For any class $[V] = [(E^0, E^1)]$ we have the additive inverse $-[V] = [(E^1, E^0)]$. We then obtain a commutative ring $K(M)$.

This leads to the following obvious result:

8.2. Lemma. *The map $M \mapsto K(M)$ is a contravariant functor from the category of manifolds to that of commutative rings. In fact, a map $f: M \to N$ induces a pull-back of vector bundles f^* and then a map $f^! : K(N) \to K(M)$ by $f^!(E^0, E^1) = (f^*E^0, f^*E^1)$.*

8.3. Characteristic classes for virtual bundles. Now let M be compact. We can extend the definition of characteristic classes from vector bundles to virtual bundles and K-theory classes: For $V = (E^0, E^1)$ we let

$$\begin{aligned} f_+(V) &= f_+(E^0) - f_+(E^1) \\ f_\times(V) &= f_\times(E^0) \wedge f_\times^{-1}(E^1). \end{aligned}$$

It follows from 7.21(c) that the definition extends to K-theory.

The fact that ch is multiplicative implies that

$$\mathrm{ch} : K(M) \to H^{\mathrm{even}}(M)$$

is a ring homomorphism.

For *noncompact* manifolds one generally does not use the concept above, but rather the following:

8.4. K-theory with compact support.

(i) A virtual bundle with compact support over the manifold M is a triple $V = (E^0, E^1, a)$, consisting of two vector bundles E^0, E^1 over M and a vector bundle morphism $a : E^0 \to E^1$ which is an isomorphism outside a compact set $X \subseteq M$. The minimal such set is the support of a.

(ii) Actually, we only need to know a outside a compact set. For this reason, one often does not bother to define a everywhere: It just needs to be given on a set with compact complement. In fact, suppose we are given an isomorphism $\tilde{a} : E^0_{|M\setminus X} \to E^1_{|M\setminus X}$ for some compact X. Choose $\rho \in C^\infty(M)$ such that $\rho = 0$ on X and $\rho = 1$ outside a compact set. Then $a = \rho\tilde{a}$ can be extended to a morphism $E^0 \to E^1$ which is an isomorphism outside a compact set. See also Lemma 8.6(a)

(iii) If M is compact, then the condition on a is void.

(iv) V is *trivial* if a is an isomorphism *everywhere.*

(v) The direct sum $V_1 \oplus V_2$ of $V_1 = (E_1^0, E_1^1, a_1)$ and $V_2 = (E_2^0, E_2^1, a_2)$ is given by

$$V_1 \oplus V_2 = (E_1^0 \oplus E_2^0, E_1^1 \oplus E_2^1, a_1 \oplus a_2).$$

(vi) Two triples V_1 and V_2 as above are *isomorphic* if there exist bundle isomorphisms

$$\phi_j : E_1^j \to E_2^j, \quad j = 0, 1,$$

which are defined everywhere on M and satisfy

(1) $$a_2 = \phi_1 a_1 \phi_0^{-1} \text{ outside a compact set.}$$

(vii) Two triples V_1 and V_2 as above are *stably isomorphic* if there exist trivial triples W_1, W_2 such that $V_1 \oplus W_1$ and $V_2 \oplus W_2$ are isomorphic. As before, we write $V_1 \sim V_2$.

8.5. The group $K_c(M)$.

(a) The equivalence classes form an abelian group with respect to direct sums: $[V_1] + [V_2] = [V_1 \oplus V_2]$. We denote this group by $K_c(M)$.

(b) $K_c(M)$ is a $K(M)$-module with the definition

$$\begin{aligned}&[(F^0, F^1)][(E^0, E^1, a)]\\ &\quad = \quad [(F^0 \otimes E^0, F^0 \otimes E^1, 1 \otimes a)] - [(F^1 \otimes E^0, F^1 \otimes E^1, 1 \otimes a)].\end{aligned}$$

Proof. (a) It is clear that the equivalence classes form a semi-group. The group property follows from Lemma 8.6, below.

(b) – □

8.6. Lemma.

(a) *Stability: For $0 \le t \le 1$ let $V(t) = (E^0, E^1, a(t))$, where $t \mapsto a(t)$ is smooth in t and $a(t)$ is an isomorphism outside some fixed compact set K (independent of t). Then $V(0)$ is isomorphic to $V(t)$ for all t. In particular $[V(0)] = [V(1)]$.*

(b) *Logarithmic property: Let $V_1 = (E^0, E^1, a)$ and $V_2 = (E^1, E^2, b)$ and let $V_3 = (E^0, E^2, ba)$. Then*

$$[V_3] = [V_1] + [V_2].$$

(c) *The inverse to $[(E^0, E^1, a)]$ is $[(E^1, E^0, a^{-1})]$.*

Proof. (a) We have to find isomorphisms $\phi^0(t) : E^0 \to E^0$ and $\phi^1(t) : E^1 \to E^1$ such that

$$a(t) = \phi^1(t) a(0) \phi^0(t)^{-1}.$$

We choose $\phi^1(t) \equiv I$, which reduces the task to finding $\phi^0(t)$ such that

$$a(t)\phi^0(t) = a(0) \text{ outside a compact set.}$$

Let ρ be a smooth function which vanishes on K and is equal to 1 outside a small neighborhood of K. For $\phi^0(t)$ we then choose a fundamental matrix for the initial value problem

$$\dot{\phi}^0 = -\rho \cdot (a^{-1})^{\cdot} a\phi^0; \quad \phi^0(0) = I;$$

(we do this for $m \in M$ in the fibers over m). We know (Analysis 2) that this system has a unique solution. On the set, where $\rho \equiv 1$, we see that $\phi(t) = a^{-1}(t)a(0)$ is a solution (hence *the* solution). This shows that the triples $V(0)$ and $V(t)$ are isomorphic: Equation 8.4(1) holds with $a_2 = a(t)$ and $a_1 = a(0)$.

(b) The triple $V_3 = (E^0, E^2, ba)$ is equivalent to the triple $(E^0 \oplus E^1, E^2 \oplus E^1, ba \oplus 1)$ which in turn is equivalent to $\left(E^0 \oplus E^1, E^1 \oplus E^2, \begin{pmatrix} 0 & 1 \\ ba & 0 \end{pmatrix}\right)$. The class of $V_1 \oplus V_2$ is, by definition, $(E^0 \oplus E^1, E^1 \oplus E^2, a \oplus b)$. Now we see that the two morphisms are homotopic, so that the triples are equivalent by part (a). In fact, consider, for $0 \le \theta \le \pi/2$,

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \cos\theta 1_{E^1} & \sin\theta 1_{E^1} \\ \sin\theta 1_{E^1} & \cos\theta 1_{E^1} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : \begin{matrix} E^0 \\ \oplus \\ E^1 \end{matrix} \to \begin{matrix} E^1 \\ \oplus \\ E^2 \end{matrix}.$$

This is a smooth family of homomorphisms (we consider the matrix in the middle as an endomorphism of $E^1 \oplus E^1$). For $\theta = 0$ we obtain $a \oplus b$, and for $\theta = \pi/2$, we obtain $\begin{pmatrix} 0 & 1 \\ ba & 0 \end{pmatrix}$.

(c) follows from (b). □

8.7. Chern character for classes in $K_c(M)$. Let $V = (E^0, E^1, a)$, and let ∂_0 and ∂_1 be connections for E^0 and E^1, respectively. By 7.9 the covariant derivative ∂b for the morphism $a : E^0 \to E^1$, is defined by

$$(\partial a)u = \partial_1(au) - a\partial_0 u.$$

We claim that we can always choose ∂_0 and ∂_1 in such a way that $\partial a = 0$ outside a compact set.

Starting with any given ∂_0 and ∂_1, let ρ be a smooth function which vanishes on K and is equal to 1 outside a small neighborhood of K. Define $\tilde{\partial}_0$ by

$$\tilde{\partial}_0 u = \partial_0 u + (\rho a^{-1}\partial a)u$$

so that the connection forms of ∂_0 and $\tilde{\partial}_0$ differ by $\delta_\Gamma = \rho a^{-1}\partial a$. For a homomorphism a, the covariant derivative $\tilde{\partial} a$ defined by $\tilde{\partial}_0$ and ∂_1 satisfies

$$\tilde{\partial} a = \partial a - a(\rho a^{-1}\partial a).$$

Hence $\tilde{\partial} a = 0$ on the set where $\rho \equiv 1$.

Note: If ∂_0 and ∂_1 are hermitian connections (i.e. E^j, $j = 0, 1$, are hermitian and $d\langle u, v\rangle = \langle \partial_j u, v\rangle + \langle u, \partial_j v\rangle$) and a is unitary, then $\tilde{\partial}_0$ is hermitian.

The curvature $\tilde{\Omega}_0$ of $\tilde{\partial}_0$ is given by

$$\tilde{\Omega}_0 = \Omega_0 + \partial(\rho a^{-1}\partial a) + (\rho a^{-1}\partial a)^2,$$

where Ω_0 is the curvature of ∂_0. We then define the form $\operatorname{ch}\xi \in H^{even}_{comp}(M)$ by

$$\operatorname{ch}\xi = \operatorname{tr} e^{\tilde{\omega}_0} - \operatorname{tr} e^{\omega_1}.$$

This form has indeed compact support: Differentiating the identity $\tilde{\partial} a = 0$ we obtain that

$$0 = \tilde{\partial}(\tilde{\partial} a) = \Omega_1 a - a\tilde{\Omega}_0.$$

Outside K we therefore have $\tilde{\Omega}_0 = a^{-1}\Omega_1 a$, so that the traces of $e^{\tilde{\omega}_0}$ and $e^{\tilde{\omega}_1}$ coincide.

It remains to check that the class is independent of the choices made.

8.8. Remark. $H^{even}_{comp}(M)$ is a $H^{even}(M)$-module and

$$ch : K_c(M) \to H^{even}_{comp}(M)$$

is a module homomorphism.

The Thom isomorphisms. Let M be a compact manifold of dimension n and E a complex vector bundle over M of rank m with a hermitian form $\langle \cdot, \cdot \rangle$. We denote by N the total space of E. This is a manifold of real dimension $n + 2m$. Writing the elements of N as pairs (x, e) with $x \in M$ and and $e \in E_x$, we define the maps

$$i : M \to N; \qquad i(x) = (x, 0)$$
$$p : N \to M; \qquad p(x, e) = x.$$

We then obtain induced maps in cohomology and K-theory

$$i^* : H^\bullet(N) \to H^\bullet(M) \text{ and } p^* : H^\bullet(M) \to H^\bullet(N)$$
$$i^! : K(N) \to K(M) \text{ and } p^! : K(M) \to K(N).$$

8.9. Lemma. *i^* and p^* and $i^!$ and $p^!$ are mutually inverse isomorphisms.*

As a consequence, $K_c(N)$ can be seen as a module over $K(M) \cong K(N)$ and $H^\bullet_{comp}(N)$ can be seen as a module over $H^\bullet(M) \cong H^\bullet(N)$.

Proof. We note that $pi = I$ and ip is homotopic to the identity: Indeed, consider the map $f_t : N \to N$, given by $f_t(x, e) = (x, te)$, $0 \le t \le 1$). Then $f_0(x, e) = (x, 0) = ip(x, e)$ and $f_1(x, e) = (x, e)$. We therefore have $p^*i^* = I$ and $i^*p^* = I$ and $p^!i^! = I, i^!p^! = I$. □

8.10. Thom Isomorphism Theorem. As a module over $K(M)$, $K_c(N)$ is generated by a single element $\beta_E \in K_c(N)$, called the Bott generator , i.e. for every $[\xi] \in K_c(N)$ there exists an element $[V] \in K(M)$ such that $[\xi] = \beta_E[p^! V]$. The theorem is usually stated that the map

$$i_! : K(M) \to K_c(N), \quad [V] \mapsto \beta_E(p^![V])$$

is an isomorphism.

Similarly, $H^\bullet_{comp}(N)$ as a $H^\bullet(M)$ module has a single generator, the so-called Thom generator U_E, and $i_!\omega = U_E \wedge p^!(\omega)$.

Proof. For details see [8, p.177f]. (Idea for the K-theoretic part) Let $\tilde{E}^* = p^!(E^*)$ be the pull-back of the dual bundle E^* of E to N. We consider the complex

$$0 \to \Lambda^0(\tilde{E}^*) \xrightarrow{\varepsilon} \Lambda^1(\tilde{E}^*) \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} \Lambda^0(\tilde{E}^*) \to 0,$$

where ε is defined as follows: The fiber $\tilde{E}_{(x,z)}$ of $\tilde{E}^*$ over the point (x, z) in N is E^*_x. By assumption E has a hermitian inner product $\langle \cdot, \cdot \rangle$. For $z \in E_x$, we obtain an element $\langle \cdot, z\rangle_x \in E^*_x$. The map $\varepsilon(z) : \Lambda^k(E^*_x) \to \Lambda^{k+1}(E^*_x)$ is given by $\varepsilon(z)(u(x, z)) = \langle \cdot, z\rangle_x \wedge u(x, z)$ for $u \in \Lambda^k(\tilde{E}^*)$.

Similarly as for the de Rham complex we see that this complex is exact outside the zero section. Just like there we consider the operator

$$b = \varepsilon + \varepsilon^* : \Lambda^{even}(\tilde{E}^*) \ \to \Lambda^{odd}(\tilde{E}^*).$$

and let

$$\beta_E = (\Lambda^{even}(\tilde{E}^*), \Lambda^{odd}(\tilde{E}^*), b) \in K_c(N).$$

For $(x, z) \in N$, the map $\varepsilon^*(z)$ is just the contraction map i_z with z:

$$i_z : \Lambda^k(\tilde{E}^*_{x,z}) \to \Lambda^{k-1}(\tilde{E}^*_{x,z}); \quad i_z u(z_1, \dots, z_{k-1}) = u(z, z_1, \dots, z_{k-1}).$$

We note that i_z satisfies

$$i_z(u \wedge v) = (i_z u) \wedge v + (-1)^k u \wedge (i_z v).$$

This implies that

$$b(z)^2 = i_z\varepsilon(z) + \varepsilon(z)i_z = |z|^2$$

the operator of multiplication by the scalar $|z|^2$. In particular, b is an isomorphism outside the zero section in E^*, which is a compact set.

In order to see that β_E is indeed the generator of $K_c(N)$, one needs additionally the Bott periodicity theorem.

(Sketch cohomological part) Let ψ be a closed form with compact support on N. Note that $\psi = \sum_J \psi_J dx^J$ with the variables $x \in M$. and fors ψ_J on E. $z \in E^*_x$. We define $p_* : \Omega^\bullet_{comp}(N) \to \Lambda^\bullet(M)$ by integrating out the variables in E_x:

$$p_*(\psi)(x) = \left(\int_{E_x} \psi_J \right) dx^I.$$

Note that E_x is of complex dimension m; as a real manifold, the dimension is $2m$. The orientation for E_x is chosen by using variables $z_1, \dots, z_m$ and $\bar{z}_1, \dots, \bar{z}_m$ and choosing $\int i^m dz^1 \wedge d\bar{z}^1 \wedge \dots dz^m \wedge d\bar{z}^m$ as positive. Of course, the integral is zero whenever $|J| \neq 2m$.

Thom's theorem states that p_* is an isomorphism. Denote by i_* its inverse. Then $U_E = i_*(1) \in H^{2m}_{comp}(N)$ is called the Thom generator. There is an explicit construction for U_E. □