7. Connections and Characteristic Classes

7.1. Notation. In this section let E be a complex vector bundle of rank N over the n-dimensional manifold M and let

$$\Lambda^p = \Lambda^p T^* M$$

be the space of alternating p-linear maps on TM, $1 \le p \le n$ (in principle, the definition extends to all $p \in \mathbb{N}_0$, but the space equals $\{0\}$ for $p = 0$ or for $p > n$). Write

$$\begin{aligned} \Lambda^\bullet &= \bigoplus_p \Lambda^p \\ \Lambda^{\text{even}} &= \bigoplus_p \Lambda^{2p} \\ \Lambda^{\text{odd}} &= \bigoplus_p \Lambda^{2p+1}; \end{aligned}$$

We will also use the bundles $E \otimes \Lambda^p$ and $\mathrm{Hom}(E, E) \otimes \Lambda^p$.[14]

Let $M_j \subseteq M$ be a coordinate neighborhood over which E is trivial (i.e. $E_{|M_j} \cong M_j \times \mathbb{C}^N$ for suitable N. Then a section $u \in C^\infty(M, E \otimes \Lambda^p)$ is locally of the form

$$u = (u^1, \ldots, u^N), \quad u^i \in \Omega^p(M),$$

and a section $a \in C^\infty(M, \mathrm{Hom}(E, E) \otimes \Lambda^q)$ locally is a matrix

$$a = (a_{jk})_{j,k=1,\ldots,N}, \quad a_{jk} \in \Omega^p(M).$$

We can then multiply $a \in C^\infty(M, \mathrm{Hom}(E, E) \otimes \Lambda^p)$ and $u \in C^\infty(M, E \otimes \Lambda^p)$ by the usual rules; the product of the above element a with u yields an element $au \in C^\infty(M, \mathrm{Hom}(E, E) \otimes \Lambda^{p+q})$.

7.2. Traces and more. The trace of an element $a \in C^\infty(M, \mathrm{Hom}(E, E) \otimes \Lambda^p)$ is taken for the matrix representing a in local coordinates: Writing $a(x) = (\sum_I a_{jk,I}(x) dx^I)_{i,j}$, we define

$$\mathrm{tr}\, a(x) = \sum_{j=1}^N \sum_I a_{jj,I}(x) dx^I.$$

This is indeed well-defined: Under a change of coordinates on M, the coefficient matrix $(a_{jk,I})$ goes over to a matrix of the form $\chi^{-1}(a_{jk,I})\chi$, where χ is an invertible matrix induced by the derivative of the change of coordinates. Hence the trace stays the same.

More generally, we may consider the characteristic polynomial of an $N \times N$-matrix

$$f_A(\lambda) = \det(\lambda I - A) = \sum_{k=0}^N c_k(A) \lambda^{n-k}.$$

[14] The tensor product $E^1 \otimes E^2$ of two vector bundles E^1 and E^2 over a compact space (here: manifold) M is defined as follows: As a set it is the disjoint union $\dot{\bigcup}_{x \in M} E^1_x \otimes E^2_x$ with the associated base point projection. For the definition of the topology see [1, §1.2].

The characteristic polynomial is invariant under conjugation of A by an invertible matrix:

$$\det(\lambda I - B^{-1}AB) = \det(B^{-1}(\lambda I - A)B) = \det(\lambda I - A).$$

For $a \in C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^{even})$ we may therefore also define the expressions $c_k(a)$ by computing them in local coordinates. We restrict to forms of even degree, since then the product in the entries is commutative. Note: $c_1 = \mathrm{tr}$, $c_N = \det$.

7.3. Graded commutators of form-valued endomorphisms. For two homomorphisms $a \in C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^p)$ and $b \in C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^q)$ we define the graded commutator

$$[a,b] := a \wedge b - (-1)^{pq} b \wedge a \in C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^{p+q}).$$

The definition extends to non-homogeneous forms (i.e. of various degrees p) by linearity. The usual trace property ($\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for matrices) then implies that

$$\mathrm{tr}[a,b] = 0,$$

the sign comes simply from rearranging the dx^j.

7.4. Connections. A connection in E is a first order differential operator $\partial : C^\infty(M,E) \to C^\infty(M, E \otimes \Lambda)$ satisfying Leibniz' rule:

$$\partial(fu) = df u + f \partial u, \quad f \in C^\infty(M), u \in C^\infty(M,E).$$

The section ∂u is called the covariant derivative of u.

7.5. Local connection 1-forms. It is sufficient to know the connection locally. In a coordinate neighborhood we can therefore choose a local frame $e = (e_1, \ldots, e_N)$, i.e. sections $e_j \in C^\infty(M,E)$ such that $\{e_1(x), \ldots, e_N(x)\}$ form a basis for the fiber E_x at every point. We can then write

$$\partial e_j = \sum_k \Gamma_j^k e_k$$

with $\Gamma_j^k \in C^\infty(M, \Lambda^1)$. The matrix $\Gamma = (\Gamma_j^k)$ is the so-called local connection 1-form; the above equation then becomes $\partial e = e\Gamma$.

Conversely, if one knows the connection 1-form then one can compute ∂u for any section $u \in C^\infty(M,E)$ with the help of Leibniz' rule in 7.4. In fact, write $u = (u^1, \ldots, u^N)$ as a column vector, so that $u = \sum e_j u^j$, then

$$\begin{aligned} \partial u &= \sum_j \partial e_j u^j + \sum_j e_j du^j \\ &= \sum_{j,k} e_k \Gamma_j^k u^j + \sum_j e_j du^j = \sum_k e_k (du^k + \sum_j u^j \Gamma_j^k). \end{aligned} \tag{1}$$

The local connection 1-forms do *not* yield a global section $\Gamma \in C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^1)$. This is a consequence of their behavior under coordinate transforms. In fact, on the intersection of two coordinate neighborhoods M_+ and M_- with local frames $e^+ = (e_1^+, \ldots, e_N^+)$ and $e^-(e_1^-, \ldots, e_N^-)$ we have $e^+ = e^- f$ for an invertible matrix-valued function f. Denoting by $\Gamma^\pm$ the associated local connection 1-forms, we see that

$$e^+\Gamma^+ = \partial e^+ = \partial(e^- f) = \partial e^- f + e^- df = e^-(\Gamma^- f + df) = e^+ f^{-1}(df + \Gamma^- f)$$

and therefore

$$\Gamma^+ = f^{-1}df + f^{-1}\Gamma^- f. \tag{2}$$

If Γ were to patch to a section of $C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^1)$ we would have $\Gamma^+ = f^{-1}\Gamma^- f$ only.

7.6. Theorem.

(a) *There exist connections for every vector bundle.*

(b) *Any two connections on a vector bundle differ by a global 1-form* $\delta_\Gamma \in C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^1)$.

Proof. (a) We lift the exterior derivative: Choose coordinate maps κ_j and a subordinate partition of unity φ_j. Moreover, choose ψ_j supported in the same chart with $\psi_j\varphi_j = \varphi_j$. Given a section $u \in C^\infty(M,E)$ consider $(\kappa_j)_*(\psi_j u)$ as a column vector on an open subset of $\mathbb{R}^n$. Let

$$\partial u = \sum \varphi_j \kappa_j^* d(\psi_j u).$$

(b) This follows from 7.5(2): In the difference of the connection 1-forms, the term that prevents the invariance $f^{-1}df$ cancels. □

7.7. Levi-Civita connection. There is an alternative way of constructing connections. Every complex vector bundle can be written as the image of a projector function[15] $P : M \to \mathcal{L}(\mathbb{C}^L)$, i.e. $E_x = \mathrm{im}\, P(x)$ for suitably large L (more when we speak about K-theory).

A section in $C^\infty(M,E)$ can be identified with a $\mathbb{C}^L$-valued function on M such that

$$P(x)u(x) = u(x), \quad x \in M. \tag{1}$$

Its exterior derivative du is a section in $C^\infty(M, \mathbb{C}^L \otimes \Lambda^1)$, but it will, in general, not satisfy (1). However, one obtains a connection – the so-called Levi-Civita connection - by setting

$$\partial u = P\, du. \tag{2}$$

7.8. Geometric interpretation. Let e_0 be a point of E, and let $x_0 = \pi e_0$ be its base point in M. The tangent space $T_{e_0}E$ of E at e_0 has a natural vertical subspace $T^v_{e_0}E$, consisting of all vectors which are tangent to the fiber E_{x_0} through e_0.

A connection allows us to find a natural complement to $T^v_{e_0}E$ of so-called horizontal vectors with the help of parallel transport along a curve. This is done as follows.

Let $\gamma : [0,\infty) \to M$ be a curve in M with $\gamma(0) = x_0$ and denote by $X(\gamma(t)) = \dot\gamma(t) \in T_{\gamma(t)}M$ the tangent vector along the curve in $\gamma(t)$. Given a section $u \in C^\infty(M,E)$, we define the covariant derivative of u in the direction X by $\partial_X u := \iota(X)\partial u := (\partial u)(X)$ (one-form applied to vector field). We call u parallel along γ, if

$$\partial_X u \equiv 0.$$

In a local frame this results in an ordinary differential equation

$$\dot u + (\iota(X)\Gamma)u = 0.$$

[15] i.e. $P = P^* = P^2$

Derivation. Fix a local frame $e = (e_1, \dots, e_N)$, write $u = \sum e_j u^j$. By 7.5(1) $\partial u = e(du + \Gamma u)$ we have $0 = \partial_X u = e(\iota(X)du + \iota(X)\Gamma u)$. Now $X = d\gamma(d/dt)$, so that

$$\iota(X)du = du(X) = dud\gamma(d/dt) = d/dt(u \circ \gamma).$$

This gives the above ODE.

It has a unique solution for each initial value $e_0 = u(x_0)$, and the solution exists for all times, since the equation is linear. We therefore obtain a lift of the curve γ to a curve in E.

This also furnishes the horizontal subspace $T^h_{e_0}E$ of $T_{e_0}E$ mentioned above: It consists of all tangent vectors of lifted curves in M through x_0. Then $\dim T^h_{e_0}E = \dim T_{x_0}M$ and hence

$$T_{e_0}E = T^v_{e_0}E \oplus T^h_{e_0}E.$$

7.9. Extending the connection to forms. The connection $\partial : C^\infty(M, E) \to C^\infty(M, E \otimes \Lambda^1)$ has a canonical extension to a map

$$\partial : C^\infty(M, E \otimes \Lambda^k) \to C^\infty(M, E \otimes \Lambda^{k+1}).$$

In fact, for $\omega \in \Omega^k$, $u \in C^\infty(M, E)$ we let

$$\partial(\omega \wedge u) = d\omega \wedge u + (-1)^k \omega \wedge \partial u.$$

In a local frame, this equals

$$d\omega \wedge u + (-1)^k \omega \wedge (du + \Gamma u) = d(\omega \wedge u) + \Gamma \wedge (\omega \wedge u).$$

Moreover, we can define ∂A for $A \in C^\infty(M, \mathrm{Hom}(E, E) \otimes \Lambda^k)$ by

$$(\partial A)u = \partial(Au) - (-1)^k A\partial u, \quad u \in C^\infty(M, E).$$

In a local frame we have

$$\begin{aligned}
(\partial A)u &= \partial(A \wedge u) - (-1)^k A \wedge \partial u \\
&= d(A \wedge u) + \Gamma \wedge (A \wedge u) - (-1)^k A(du + \Gamma \wedge u) \\
&= dA \wedge u + (-1)^k A \wedge du + \Gamma \wedge A \wedge u \\
&\quad -(-1)^k A \wedge du - (-1)^k A \wedge \Gamma \wedge u \\
&= (dA + [\Gamma, A])u. \qquad (1)
\end{aligned}$$

Even more generally: If E_0 and E_1 are bundles over M with connections ∂_0 and ∂_1, then one defines for $A \in C^\infty(M, \mathrm{Hom}(E_0, E_1) \otimes \Lambda^k)$

$$(\partial A)u = \partial_1(Au) - (-1)^k A\partial_0 u. \qquad (2)$$

7.10. Traces. For $A \in C^\infty(M, \mathrm{Hom}(E, E) \otimes \Lambda^k)$ we define the trace $\mathrm{tr}\, A \in C^\infty(M, \Lambda^k)$ by taking the matrix trace of its representation in local coordinates. This is independent of the choice of coordinates by the considerations in 7.1. Since the trace vanishes on graded commutators of forms, we find that

$$\mathrm{tr}\, \partial A \overset{7.9(1)}{=} \mathrm{tr}(dA + [\Gamma, A]) = \mathrm{tr}\, dA = d\, \mathrm{tr}\, A.$$

7.11. Curvature. There is a central observation: The square ∂^2 of the curvature is not – as one might expect – a second order operator, but of order zero, an endomorphism. In fact,

$$\partial^2 = \partial\partial : C^\infty(M, E \otimes \Lambda^l) \to C^\infty(M, E \otimes \Lambda^{k+2})$$

is given by

$$\partial^2 u = \Omega u, \quad \Omega \in C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^2);$$

Ω is called the curvature of ∂. In order to see this we note that in a local frame

$$\begin{aligned}\partial^2 u = d\partial u + \Gamma \wedge \partial u \\ &= d(du + \Gamma u) + \Gamma \wedge (du + \Gamma u) \\ &= 0 + d\Gamma u - \Gamma du + \Gamma du + \Gamma \wedge \Gamma u \\ &= (d\Gamma + \Gamma \wedge \Gamma)u.\end{aligned}$$

In each frame we therefore have

$$\Omega = d\Gamma + \Gamma \wedge \Gamma = d\Gamma + \frac{1}{2}[\Gamma, \Gamma].$$

In fact, the local forms on the right hand side patch here to a global 2-form. In order to see this we note that $0 = d(Id) = d(f^{-1}f) = d(f^{-1})f + f^{-1}df$ so that $d(f^{-1})f = -f^{-1}df$. We conclude from the change-of-coordinates formula for the connection 1-form: $\Gamma^+ = f^{-1}\Gamma^- f + f^{-1}df$ that

$$d\Gamma^+ = d(f^{-1}\Gamma^- f) + d(f^{-1})df + 0 = d(f^{-1}\Gamma^- f) - (f^{-1}df)^2.$$

On the other hand,

$$\begin{aligned}&\Gamma^+ \wedge \Gamma^+ \\ &= (f^{-1}\Gamma^- f)^2 + f^{-1}\Gamma^- f \wedge f^{-1}df + f^{-1}df \wedge f^{-1}\Gamma^- f + (f^{-1}df)^2 \\ &= f^{-1}\Gamma^- \wedge \Gamma^- f + (f^{-1}df)^2.\end{aligned}$$

Hence

$$d\Gamma^+ + \Gamma^+ \wedge \Gamma^+ = f^{-1}(d\Gamma^- + \Gamma^- \wedge \Gamma^-)f$$

so that the non-invariant terms cancel.

7.12. Lemma.

(a) $\partial^2 A = [\Omega, A]$ *for* $A \in C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^k)$ *(graded commutator).*

(b) $\partial^2 A = \Omega_1 \wedge A - A \wedge \Omega_0$ *for* $A \in C^\infty(M, \mathrm{Hom}(E_0, E_1) \otimes \Lambda^k)$, *where* Ω_0 *and* Ω_1 *are the curvatures of the connections on* E_0 *and* E_1, *respectively.*

Proof. We only have to show (b):

$$\begin{aligned}(\partial^2 A)u = (\partial(\partial A))u &\overset{7.9(2)}{=} \partial_1(\partial A)u - (-1)^k(\partial A)\partial_0 u \\ &= \partial_1(\partial_1 Au - (-1)^k A\partial_0 u) - (-1)^{k+1}(\partial_1(A\partial_0 u) - (-1)^k A\partial_0^2 u \\ &= \Omega_1 Au - A\Omega_0 u.\end{aligned}$$

□

7.13. Definition. A connection ∂ is called *flat* if $\Omega = 0$ and abelian, if $\Omega = c\,\mathrm{Id}_E$ for a scalar form c.

7.14. Curvature of the Levi-Civita connection. Consider the Levi-Civita connection of 7.7. There

$$\partial^2 u = Pd(\partial u) = Pd(Pdu) = PdP \wedge du + 0$$

Moreover, since u is a section, i.e. $u = Pu$, we have $du = d(Pu) = dP \wedge u + P \wedge du$ and thus

$$\begin{aligned} \partial^2 u &= PdP \wedge (dP \wedge u + P \wedge du) \\ &= (PdP \wedge dP)u + PdPP \wedge du = (PdP \wedge dP)u, \end{aligned}$$

note that the term $P\,dP\,P$ is zero, since P is a projection: this follows form multiplying the identity $dP = d(PP) = dPP + PdP$ by P.

7.15. Lemma.

(a) *If ∂ is abelian, then $\partial^2 A = 0$ for every $A \in C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^k)$*

(b) *If E is trivial with a trivial frame, then there exists a flat connection in E*

(c) *For E of rank 1, every connection is abelian.*

Proof. (a) follows from 7.12(a). (b) Choose $\Gamma = 0$. (c) Here Γ is 1×1. □

7.16. Bianchi identity. $\partial\Omega = 0$.

Indeed, for $u \in C^\infty(M, E)$ they follows from the fact that on one hand,

$$\partial^3 u = \partial(\partial^2 u) = \partial(\Omega u) = \partial\Omega \wedge u + \Omega \wedge \partial u;$$

on the other

$$\partial^3 u = \partial^2(\partial u) = \Omega \wedge \partial u.$$

7.17. Variational formula. Suppose we are given a smooth family of connections $\{\partial_{(t)} : 0 \le t \le 1\}$ with connection 1-forms Γ_t. We can write

$$\partial_{(t)} u = d + \Gamma_t = \partial_{(0)} + \delta_\Gamma(t),$$

where, as we know form Theorem 7.6, the difference $\delta_\Gamma(t) = \Gamma_t - \Gamma_0$ is a global 1-form. For $w \in C([0,1], C^\infty(M,E))$ and $\dot f = df/dt$,

$$\begin{aligned} (\partial_{(t)} w)^{\cdot} &= (\partial_{(0)} w + \delta_\Gamma w)^{\cdot} \\ &= \partial_{(0)}\dot w + \dot\delta_\Gamma w + \delta_\Gamma \dot w = \partial_{(t)}\dot w + \dot\delta_{\Gamma_t} w = \partial_{(t)}\dot w + \dot\Gamma_t w \qquad (1) \end{aligned}$$

In particular, $\dot\Gamma$ is a (globally defined) section in $C^\infty(M, \mathrm{Hom}(E,E) \otimes \Lambda^1)$. If u is actually independent of t, then

$$(\partial_{(t)} u)^{\cdot} = (\delta\Gamma)^{\cdot}(t)u = \dot\Gamma_t u,$$

Moreover, (1) implies (for u independent of t and $w = \partial_{(t)} u$))

$$(\Omega(t)u)^{\cdot} = (\partial_{(t)}(\partial_{(t)} u))^{\cdot} = \partial_{(t)}(\Gamma_t u)^{\cdot} + \dot\Gamma_t \partial_{(t)} u$$

Now $(\Gamma_t u)^{\cdot} = \dot\Gamma_t u$ and therefore – according to the rules for connections, since $\dot\Gamma$ is a one-form –

$$(\Omega(t)u)^{\cdot} = \partial_{(t)}(\dot\Gamma_t u) = \partial_{(t)}(\dot\Gamma_t)u - \dot\Gamma_t \partial_{(t)} u + \dot\Gamma_t \partial_{(t)} u = \partial_{(t)}\dot\Gamma_t u$$

so that

$$\dot\Omega = \partial_{(t)}\dot\Gamma_t.$$

7.18. Adams forms. Let ∂ be a connection for E and Ω its curvature. Define $\omega = -\frac{\Omega}{2\pi i}$. Then ω^k is a $\mathrm{Hom}(E,E)$-valued $2k$-form. We define the Adams forms

$$\psi_k = \operatorname{tr}\omega^k \in \Omega^{2k}(M).$$

These are closed forms, since

$$d\psi_k = d(\operatorname{tr}\omega^k) \overset{7.21}{=} \operatorname{tr}(\partial\omega^k) \overset{\text{Leibniz, even}}{=} k\operatorname{tr}\omega^{k-1}\partial\omega \overset{\text{Bianchi (7.16), } \partial\omega=0}{=} 0.$$

we can therefore consider the ψ_k as elements of $H^{2k}(M)$. Under this point of view, the Adams forms are independent of the connection we started with: Let ∂_0 and ∂_1 be connections and let

$$\partial_{(t)} = (1-t)\partial_0 + t\partial_1.$$

The variational formula for the associated connection $\Omega(t)$ then implies

$$\begin{aligned}\frac{d}{dt}\operatorname{tr}(\Omega^k(t))^k &\overset{\text{even form}}{=} k\operatorname{tr}(\Omega^{(k-1)}\wedge\dot\Omega)\\ &\overset{7.17}{=} k\operatorname{tr}(\Omega^{(k-1)}\wedge\partial_{(t)}\dot\Gamma \overset{\partial_{(t)}\Omega=0}{=} k\operatorname{tr}\partial_{(t)}(\Omega^{(k-1)}\wedge\dot\Gamma)\\ &= kd\operatorname{tr}(\Omega^{k-1}\wedge\dot\Gamma),\end{aligned}$$

which is an exact form.

7.19. Chern forms. As before let ∂ be a connection for E and Ω its curvature. The Chern forms are the coefficients $c_k \in \Omega^{2k}$ of the characteristic polynomial, cf. 7.2:

$$\det(\lambda - \omega) = \sum_{k=0}^{N} \lambda^{N-k} c_k.$$

The c_k are polynomials in the Adams forms ψ_k and conversely. This is a consequence of Newton's identities which express the elementary symmetric polynomials e_k in the eigenvalues $\lambda_1,\ldots,\lambda_N$ of a matrix in terms of the functions $p_k = \lambda_1^k + \ldots + \Lambda_N^k$, $k = 0,\ldots,N$. The formula then hold in every commutative ring.

$$\begin{aligned}
&e_0 = 1 && \\
&e_1 = \sum \lambda_j && \text{hence } e_1 = p_1\\
&e_2 = \sum_{j<k} \lambda_j\lambda_k && \text{hence } 2e_2 = e_1p_1 - p_2\\
&e_3 = \sum_{j<k<l} \lambda_j\lambda_k\lambda_l && \text{hence } 3e_3 = e_2p_1 - e_1p_2 + p_3\\
&\vdots && \\
&e_n = \lambda_1\cdots\lambda_N && \text{hence } Ne_N = e_{N-1}p_1 - e_{N-2}p_2 + \ldots \pm p_N
\end{aligned}$$

As a consequence, also the c_k are closed forms by Leibniz' rule (note that they are of even degree), and we can consider c_k as an element of $H^{2k}(M)$. In this respect, also c_k is independent of the choice of the connection. We call the c_k the Chern characteristic classes. One often writes $c_k(E)$ to express the dependence on E. The sum

$$C(E) = 1 + c_1(E) + c_N(E) \in \Omega^{2N}(M)$$

is called the complete Chern class of E.

7.20. More general classes. Let $f(\lambda) = \sum_k a_k \lambda^k$ be a formal power series with coefficients in $\mathbb{C}$. Then we can define the non-homogeneous even cohomology classes

$$\begin{aligned} f_+(E) &= \operatorname{tr}\big(f(\omega)\big) \quad \text{(additive characteristic classes);} \\ f_\times(E) &= \det\big(f(\omega)\big) \quad \text{(multiplicative characteristic classes).} \end{aligned}$$

Note that there is no problem of convergence, since M is finite-dimensional, so that $\omega^k = 0$ for large k. All these classes can be expressed in terms of Adams forms or Chern forms; they are therefore independent of the choice of the connection.

Examples.

$C(E) = 1 + c_1(E) + \ldots + c_N(E) = \det(\lambda - \omega)$ the complete Chern class
$\operatorname{ch}(E) = \operatorname{tr}(e^\omega)$ the Chern character
$\operatorname{Td}(E) = \det(\omega/(1 - \exp(-\omega)))$ the Todd class
$\widehat{A}(E) = \det\Big(((\omega/2)/\sinh(\omega/2))^{1/2}\Big)$ the Atiyah-Hirzebruch class

7.21. Simple properties of characteristic classes.

(a) If E is flat or ∂ is abelian, then $\Omega = 0$ and only the constant term survives in the power series. In particular $C(E) = 1$ and $\operatorname{ch}(E) = 1$.

(b) If $\varphi : M \to N$ is a smooth mapping between the manifolds M and N and E a vector bundle over N, then φ induces on one hand a bundle $\varphi^* E$ over M and on the other a map $\varphi^* : H^j(N) \to H^j(M)$ of the cohomology classes. For every characteristic class we then have

$$f(\varphi^* E) = \varphi^* f(E).$$

(c) $f_+(E_1 \oplus E_2) = f_+(E_1) + f_+(E_2)$ for an additive characteristic class and bundles E_1, E_2 over M, while $f_\times(E_1 \oplus E_2) = f_\times(E_1) \wedge f_\times(E_2)$ for a multiplicative class.

(d) Let E^* be the dual bundle to E, whose fiber at $x \in M$ is the dual space $(E_x)^*$. Then $f(E^*) = g(E)$, where $g(\lambda) = f(-\lambda)$.

Proof. (a) is obvious.

(b) On M we can use the pull-back connection $\varphi^*\partial$, which is defined by the property that

$$((\varphi^*\partial)(\varphi^* s))(X) = \varphi^*(\partial s)(d\varphi X)$$

with curvature $\varphi^*\Omega$.

(c) For the associated connections we have

$$\operatorname{tr}_{E_1 \oplus E_2}(\Omega_1 \oplus \Omega_2) = \operatorname{tr}_{E_1}(\Omega_1) + \operatorname{tr}_{E_2}(\Omega_2)$$

and

$$\det_{E_1 \oplus E_2}(\Omega_1 \oplus \Omega_2) = \det_{E_1}(\Omega_1) \wedge \det_{E_2}(\Omega_2).$$

(d) Replacing Γ by $-\Gamma^t$, we obtain a connection for the dual bundle E^*. Its curvature is

$$\begin{aligned}\Omega^* &= d(-\Gamma^t) + (-\Gamma^t) \wedge (-\Gamma^t) = -(d\Gamma)^t + \Gamma^t \wedge \Gamma^t \\ &\overset{\text{1-forms}}{=} \quad -d\Gamma^t - (\Gamma \wedge \Gamma)^t = -\Omega.\end{aligned}$$

Now the definition of the classes yields the assertion. □

7.22. Properties of the Chern character.

(a) Let E_1 and E_2 be complex vector bundles over M. Then $\operatorname{ch}(E_1 \otimes E_2) = \operatorname{ch}(E_1) \wedge \operatorname{ch}(E_2)$.

(b) For a E as above, we define the k-th exterior power $\Lambda^k(E)$ as the subbundle of all antisymmetric tensors in $\bigotimes^k E$. Then

$$\operatorname{ch}(\Lambda^k(E)) = \operatorname{ch}(\Lambda^k e^\omega) =: \operatorname{ch}(e^\omega \wedge \ldots \wedge e^\omega) \quad (k \text{ factors}).$$

(c) By $\Lambda^\pm(E^*)$ we denote the sums of all even/odd exterior powers for E^*. Then

$$\operatorname{ch}(\Lambda^+(E^*)) - \operatorname{ch}(\Lambda^-(E^*)) = c_N(E)\operatorname{Td}^{-1}(E).$$

On the right hand side, Td^{-1} is the inverse power series to Td with respect to multiplication.

Proof. (a) Given the connections ∂_j on E_j, $j = 1, 2$, define a connection ∂ on $E_1 \otimes E_2$ by

$$\partial = \partial_1 \otimes 1 + 1 \otimes \partial_2,$$

i.e. $\partial(u_1 \otimes u_2) = (\partial_1 u_1) \otimes u_2 + u_1 \otimes (\partial_2 u_2)$. For the curvature Ω we find from the rules for the application of connections on forms 7.9:

$$\begin{aligned}\Omega(u_1 \otimes u_2) &= \partial^2(u_1 \otimes u_2) = \partial((\partial_1 u_1) \otimes u_2 + u_1 \otimes (\partial_2 u_2)) \\ &= (\partial_1^2 u_1) \otimes u_2 - (\partial_1 u_1) \otimes (\partial_2 u_2) + (\partial_1 u_1) \otimes (\partial_2 u_2) + u_1 \otimes (\partial_2^2 u_2) \\ &= (\Omega_1 u_1) \otimes u_2 + u_1 \otimes (\Omega_2 u_2)\end{aligned}$$

so that $\Omega = \Omega_1 \otimes 1 + 1 \otimes \Omega_2$. Since the summands commute, we conclude that

$$\begin{aligned}\exp(\omega) &= \exp(1 \otimes \omega_1 + 1 \otimes \omega_2) \\ &= \exp(1 \otimes \omega_1)\exp(1 \otimes \omega_2) = \exp(\omega_1)\exp(\omega_2) \qquad (1)\end{aligned}$$

Taking traces yields the assertion.

(b) We first apply (1) to the normalized curvature of the k-fold tensor product $\bigotimes^k E$ and conclude that, for the associated 2-form ω_k:

$$\exp(\omega_k) = \exp\omega \wedge \ldots \wedge \exp\omega \quad (k \text{ times}). \qquad (2)$$

We then project onto the space of anti-symmetric tensors and take the trace to obtain the assertion.

(c) We use the following result from linear algebra: For $A \in \mathcal{L}(V)$, where V is a finite-dimensional vector space over $\mathbb{C}$ we have

$$\det(I - A) = \sum (-1)^k \operatorname{tr}(\Lambda^k A),$$

where $\Lambda^k A$ is $A \otimes \ldots \otimes A$ applied to the anti-symmetric tensors.

We conclude that

$$\mathrm{ch}(\Lambda^+(E)) - \mathrm{ch}(\Lambda^-(E)) = \sum_k (-1)^k \,\mathrm{tr}(\Lambda^k \exp(\omega)) = \det(I - \exp(\omega)).$$

Replacing E by E^* and using that $\mathrm{ch}\, f(E^*) = \mathrm{ch}\, g(E)$, where $g\lambda) = f(-\lambda)$, we see that

$$\begin{aligned} & \mathrm{ch}(\Lambda^+(E^*)) - \mathrm{ch}(\Lambda^-(E^*)) \\ = \;& \sum_k (-1)^k \,\mathrm{tr}(\Lambda^k \exp(-\omega)) = \det(I - \exp(-\omega)) \\ = \;& \det\Big(\omega\Big(\frac{\omega}{I - e^{-\omega}}\Big)^{-1}\Big) = (\det\omega)\det\Big(\frac{\omega}{I - e^{-\omega}}\Big)^{-1} \\ = \;& c_N(E)(\mathrm{Td})^{-1}(E). \end{aligned}$$

□

7.23. Real bundles. In the above construction, E was a complex bundle. For a real bundle, we go over to its complexification and define the characteristic classes $f_+(E) = \mathrm{tr}(f(-\Omega/(2\pi i))$ and $f_\times(E) = \det(f(-\Omega/(2\pi i))$ for a power series f as before. Since in the real case, the bundles E and E^* are canonically isomorphic, we conclude from 7.21(d) that the classes for $f(\lambda)$ and $f(-\lambda)$ coincide. Hence only the terms associated with the even powers of λ appear.

In particular, all odd Chern forms c_{2k+1} are then zero. One defines in this case

$$p_k(E) = (-1)^k c_{2k}(E) \qquad k\text{-th Pontryagin class.}$$

7.24. Remark. The Pontryagin classes appeared in the index theorem for the signature complex. The L-genus is derived from the power series

$$f(\lambda) = \frac{\sqrt{\lambda}}{\tanh\sqrt{\lambda}} = \sum_{k\geq 0} 2^{2k} B_{2k} \frac{z^k}{(2k)!} = 1 + \frac{z}{3} - \frac{z^2}{45} + \cdots,$$

where the numbers B_{2k} are the Bernoulli numbers. The first few values are:

$$\begin{aligned} L_0 &= 1 \\ L_1 &= \tfrac{1}{3}p_1 \\ L_2 &= \tfrac{1}{45}(7p_2 - p_1^2) \\ L_3 &= \tfrac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3) \\ L_4 &= \tfrac{1}{14175}(381p_4 - 71p_1p_3 - 19p_2^2 + 22p_1^2p_2 - 3p_1^4) \end{aligned}$$

.

7.25. Remark. One often writes $\mathrm{Td}(M)$ when one actually means $\mathrm{Td}(TM)$. Similarly for other characteristic classes.