6. Elliptic Complexes

6.1. Definition. Let E^0, $E^1, \dots$, be vector bundles over M. A complex of pseudodifferential operators is a finite sequence

(1) $0 \to C^\infty(M,E^0) \xrightarrow{P^0} C^\infty(M,E^1) \xrightarrow{P^1} \dots \xrightarrow{P^{m-1}} C^\infty(M,E^m) \to 0$

of pseudodifferential operators P^j such that $P^{j+1}P^j = 0$, i.e. $\operatorname{im} P^j \subseteq \ker P^{j+1}$.

It will make the considerations in 6.4, below, easier, if we assume that all P^j have the same order, say m. In practically appearing cases, we mostly have $m = 1$; otherwise it can easily be achieved with the help of order reducing operators.

The complex is called elliptic, if the associated complex of symbol maps

$$0 \to \pi^*E^0 \xrightarrow{\sigma(P^0)} \pi^*E^1 \xrightarrow{\sigma(P^1)} \dots \xrightarrow{\sigma(P^{m-1})} \pi^*E^m \to 0 \qquad (2)$$

over $T^*M \setminus 0$ is exact, i.e. $\operatorname{im}\sigma(P^j) = \ker\sigma(P^{j+1})$. Here $\pi : T^*M \setminus 0 \to M$ is the map to the base point.

In the sequel, we will also assume that the vector bundles have a hermitian structure, i.e. there is a scalar product on every fiber, varying smoothly with the base point.

6.2. Example. Let $\Lambda^k = \Lambda^k T^*M$ be the bundle of k-forms over M. As usual, we write $\Omega^k M = C^\infty(M,\Lambda^k)$ Consider the de Rham complex

$$0 \to \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \dots \xrightarrow{d} \Omega^n M \to 0,$$

where $n = \dim M$. The associated symbol complex is

$$0 \to \pi^*\Lambda^0 \xrightarrow{i\xi\wedge} \pi^*\Lambda^1 \xrightarrow{i\xi\wedge} \dots \xrightarrow{I\xi\wedge} \pi^*\Lambda^n \to 0,$$

cf. Example 5.31. As $\xi \wedge \xi = 0$ for every $\xi \in T^*M \setminus 0$, we have $\operatorname{im}\sigma(d^j) \subseteq \ker\sigma(d^{j+1})$. In order to see the reverse inclusion assume that $\xi \wedge \eta = 0$ for a form $\eta = \sum f_I dx^I$ where the sum is over all $I = (i_1, \dots, i_k)$ with $1 \le i_1 < \dots < i_k \le n$. By a linear change of coordinates we may assume that $\xi = dx^1$. The fact that $\xi \wedge \eta = 0$ means that $i_1 = 1$ for all I with $f_I \neq 0$. Hence $\eta \in \operatorname{im}(dx^1\wedge)$.

6.3. Example. Every elliptic pseudodifferential operator $P : C^\infty(M,E^1) \to C^\infty(M,E^2)$ defines an elliptic complex

$$0 \to C^\infty(M,E^1) \to C^\infty(M,E^2) \to 0.$$

In fact,

$$0 \to \pi^*E^1 \xrightarrow{\sigma(P)} \pi^*E^2 \to 0$$

is exact, since $\sigma(P)(x,\xi)$ is invertible.

6.4. From Complexes to operators. To every complex we can associate operators in a canonical way: Write

$$\mathscr{E} = \bigoplus_j E^j, \mathscr{E}^{\text{even}} = \bigoplus_j E^{2j}, \mathscr{E}^{\text{odd}} = \bigoplus_j E^{2j+1}$$

and define the $(m+1)\times(m+1)$ operator matrices

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ P^0 & 0 & \dots & 0 & 0 \\ 0 & P^1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P^{m-1} & 0 \end{pmatrix}, \quad P^* = \begin{pmatrix} 0 & P^{0*} & 0 & \dots & 0 & 0 \\ 0 & 0 & P^{1*} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & P^{m-1*} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

where $P^{j*} : C^\infty(M, E^{j+1}) \to C^\infty(M, E^j)$ is the formal adjoint to P^j. The identity $P^{j+1}P^j$ implies that $P^2 = 0 = (P^*)^2$. Then define

$$\Delta = (P + P^*)^2 = P^*P + PP^*.$$

A short computation shows that

$$\Delta = \begin{pmatrix} P^{0*}P^0 & 0 & \dots & 0 & 0 \\ 0 & P^{1*}P^1 + P^0P^{0*} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P^{m-1*}P^{m-1} + P^{m-2}P^{m-2*} & 0 \\ 0 & 0 & \dots & 0 & P^{m-1}P^{m-1*} \end{pmatrix}$$

Δ is called the Hodge-Laplace operator, in particular, if the P_j are the exterior derivatives on forms.

Assuming - as we do - that all P^j are of order m, we see that Δ is a pseudodifferentialoperator of order $2m$ and therefore has continuous extensions to a bounded operator $\Delta : H^s(M, \mathscr{E}) \to H^{s-2m}(M, \mathscr{E})$.

6.5. Example. Choosing a scalar product on the fibers T^*M (which is equivalent to choosing a riemannian metric on M) we obtain scalar products on all spaces $\Omega^k(M)$, $k = 0, \dots, n$. The adjoint to the de Rham differential d is often denoted by δ. We know from the general theory that the symbol of the adjoint is the adjoint of the symbol. Here

$$\sigma(d) : \pi_0^*\Omega^k(M) \to \pi_0^*\Omega^{k+1}(M); \quad \sigma(d)(x,\xi)\eta = \xi \wedge \eta,\ \eta \in \pi_0^*\Omega^k(M).$$

The adjoint map is the interior multiplication map $\operatorname{int}(i\xi)$ with $-i\xi$ (in contrast to the map $\operatorname{ext}(i\xi) = i\xi\wedge$ of exterior multiplication with $i\xi$:

$$\sigma(\delta) = \sigma(d)^* : \pi_0^*\Omega^{k+1}(M) \to \pi_0^*\Omega^k(M).$$

The scalar product on T^*M induces, for every $x \in M$, an isomorphism $j : T_x^*M \overset{\cong}{\to} T_xM$ by

$$\eta(j(\omega)) = \langle \eta, \omega \rangle, \quad \eta, \omega \in T^*M.$$

We then set

$$\operatorname{int}(-i\xi)\eta = -i\iota_{j(\xi)}\eta, \quad \eta \in \Omega^{k+1}(M).$$

The symbol of the Laplacian on k-forms then is

$$\begin{aligned} \sigma(\Delta)(x,\xi) &= (\sigma(d)\sigma(\delta) + \sigma(\delta)\sigma(d))(x,\xi) \\ &= \operatorname{ext}(i\xi)\operatorname{int}(-i\xi) + \operatorname{int}(-i\xi)\operatorname{ext}(i\xi) = |\xi|^2 Id_{\Lambda^k}. \end{aligned}$$

6.6. Lemma. *The following are equivalent:*

(i) *The complex* 6.1(1) *is elliptic*

(ii) $\Delta : C^\infty(M, \mathscr{E}) \to C^\infty(M, \mathscr{E})$ *is elliptic*

(iii) $P + P^* : C^\infty(M, \mathscr{E}) \to C^\infty(M, \mathscr{E})$ *is elliptic.*

Proof. Since $\Delta = (P+P^*)^2$, conditions (ii) and (iii) are equivalent by Theorem 5.25.

Moreover, Δ is elliptic if and only if all diagonal elements are elliptic. Let us check that this is equivalent to the exactness of the complex

$$0 \longrightarrow \pi_0^*E^0 \overset{\sigma(P^0)}{\longrightarrow} \pi_0^*E^1 \overset{\sigma(P^1)}{\longrightarrow} \dots \overset{\sigma(P^{m-1})}{\longrightarrow} \pi_0^*E^m \longrightarrow 0. \tag{1}$$

Write $p_j = \sigma(P^j)$. First assume that (1) is exact. Then in the first place p_0 is injective, and hence $p_0^*p_0$ invertible.

In the next place, assume that $\operatorname{im} p_0 = \ker p_1$ and show the invertibility of $p_1^*p_1 + p_0p_0^*$. It suffices to prove that the kernel is trivial. Suppose $p_1^*p_1x + p_0p_0^*x = 0$. Then $\langle p_1^*p_1x, x\rangle + \langle p_0p_0^*x, x\rangle = 0$ and hence $\langle p_1x, p_1x\rangle + \langle p_0^*x, p_0^*x\rangle = 0$. We see that $x \in \ker p_1 = \operatorname{im} p_0$, so that $x = p_0y$ for some y and $p_0^*p_0y = p_0^*x = 0$. Since $p_0^*p_0$ is invertible, $y = 0$ and therefore $x = 0$. Iteration shows the invertibility of all $p_{j+1}^*p_{j+1} + p_jp_j^*$.

Conversely, suppose that Δ is elliptic. In the first place, the invertibility of $p_0^*p_0$ implies the injectivity of p_0. Next suppose $x \in \pi_0^*E^1$ belongs to the kernel of p_1. As $p_1^*p_1 + p_0p_0^*$ is invertible, we find $y \in \pi_0^*E^1$ such that $x = (p_1^*p_1 + p_0p_0^*)y$. The fact that x is in the kernel of p_1 together with the fact that $p_1p_0 = 0$ implies that $0 = p_1x = p_1p_1^*p_1y$. Hence $p_1y \in \ker p_1p_1^* = \ker p_1^*$. So $p_1^*p_1y = 0$ and thus $x = p_0p_0^*y \in \operatorname{im} p_0$. We argue analogously for the other places. □

6.7. Lemma. *Assume that the complex* 6.1(1) *is elliptic. Then*

(a) $\ker\Delta \subseteq C^\infty(M,\mathscr{E})$ *for all extensions* $\Delta : H^s(M,\mathscr{E}) \to H^{s-m}(M,\mathscr{E})$.

(b) $\ker\Delta = \ker P \cap \ker P^*$.

Proof. (a) follows from elliptic regularity.

(b) We have $\langle\Delta u, u\rangle = \langle(P^*P + PP^*)u, u\rangle = \langle Pu, Pu\rangle + \langle P^*u, P^*u\rangle$, so that $\ker\Delta \subseteq \ker P \cap \ker P^*$. Conversely, if $u \in \ker P \cap \ker P^*$, then $\Delta u = (P^*P + PP^*)u = 0$. □

6.8. Theorem: Hodge decomposition. *Assume that the complex* 6.1(1) *is elliptic. Then*

$$\begin{aligned} C^\infty(M,\mathscr{E}) &= \ker\Delta \perp \operatorname{im} P^*P_{|C^\infty(M,\mathscr{E})} \perp \operatorname{im} PP^*_{|C^\infty(M,\mathscr{E})} \\ &= \ker\Delta \perp \operatorname{im} P^*_{|C^\infty(M,\mathscr{E})} \perp \operatorname{im} P_{|C^\infty(M,\mathscr{E})}, \end{aligned}$$

where we consider P *and* P^* *as maps on* $C^\infty(M,\mathscr{E})$ *and write* $\perp$ *for the orthogonal direct sum.*

Proof. '$\supseteq$' is clear from Lemma 6.7 and the mapping properties.

'$\subseteq$': Denote, for the moment, by $\pi_{\ker\Delta}$ the orthogonal projection onto the (finite-dimensional) kernel of Δ in $L^2(M,\mathscr{E})$. Given $u \in C^\infty(M,\mathscr{E})$ let

$u_0 = \pi_{\ker \Delta} u$. Then

$$\begin{aligned} u - u_0 &\in (\ker \Delta)^\perp \cap C^\infty(M, \mathscr{E}) \\ &= \overline{\operatorname{im} \Delta^*} \cap C^\infty(M, \mathscr{E}) \\ &\overset{\Delta = \Delta^*}{=} \overline{\operatorname{im} \Delta} \cap C^\infty(M, \mathscr{E}) \\ &\overset{\Delta \text{Fredholm}}{=} \operatorname{im} \Delta \cap C^\infty(M, \mathscr{E}) \\ &= \operatorname{im} \Delta_{|C^\infty(M, \mathscr{E})} \\ &= \operatorname{im}(P^*P + PP^*) \\ &\subseteq \operatorname{im} P^*P + \operatorname{im} PP^* \\ &\subseteq \operatorname{im} P^* + \operatorname{im} P. \end{aligned}$$

Moreover, we note that the right hand side is a subset of $(\ker \Delta)^\perp$ and that the last sum is orthogonal in view of the fact that for $u, v \in C^\infty(M, \mathscr{E})$:

$$\langle Pu, P^*v \rangle = \langle P^2 u, v \rangle = 0, \text{ since } P^2 = 0 \text{ (complex!)}.$$

□

6.9. Hodge Theorem. *Assume that the complex* 6.1(1) *is elliptic and write* $\Delta = diag\ (\Delta_1, \ldots, \Delta_m)$. *Then*

$$\ker \Delta_k \cong \ker P_k / \operatorname{im} P_{k-1} =: H^k, \quad k = 1, \ldots, m.$$

Note that the quotient makes sense, since the complex property $P_k P_{k-1} = 0$ guarantees that $\operatorname{im} P_{k-1} \subseteq \ker P_k$. Since Δ is a Fredholm operator, the quotient is finite-dimensional.

Proof. We define the map

$$\ker \Delta \ni u \mapsto [u] = u + \operatorname{im} P_{k-1} \in H^k$$

According to Lemma 6.7 this makes sense.

The map is injective: Suppose $[u] = 0$. Then $u \in \operatorname{im} P_{k-1}$, which is orthogonal to $\ker \Delta$ by Theorem 6.8. Hence $u = 0$.

Surjectivity. Suppose that $u \in \ker P_k$. Apply P_k to the Hodge decomposition

$$u = u_0 + P_{k-1} v + P_k^* w$$

of u (with $u_0 \in \ker \Delta, v \in C^\infty(M, \mathscr{E}_{k-1}), w \in C^\infty(M, \mathscr{E}_k)$), we see that

$$0 = P_k u_0 + P_k P_{k-1} v + P_k P_k^* w = 0 + 0 + P_k P_k^* w$$

since $\ker \Delta \subseteq \ker P_k$ according to Lemma 6.7(b), and since $P_k P_{k-1} = 0$. We conclude that $P_k P_k^* w = 0$ and therefore $P_k^* w = 0$. Hence $[u_0] = [u]$. □

6.10. Definition. Given an elliptic complex 6.1(1) we can define the operator

$$\mathcal{P} = (P + P^*)_{|C^\infty(M, \mathscr{E}^{\text{even}})} : C^\infty(M, \mathscr{E}^{\text{even}}) \to C^\infty(M, \mathscr{E}^{\text{odd}}).$$

Moreover, one defines the index of the complex to be the index of $\mathcal{P}$. This is consistent with the usual definition by Example 6.3. Denote by p_k the symbol of P_k. To see the injectivity of $\sigma(\mathcal{P})$ we note that for $v = (v_0, v_2, \ldots)$ the equality $\sigma(\mathcal{P})v = 0$ says that

$$\begin{aligned} p_0 v_0 + p_1^* v_2 &= 0 \\ p_2 v_2 + p_3^* v_4 &= 0 \\ &\vdots \end{aligned}$$

The first equality implies that $p_1^* v_2 \in \operatorname{im} p_0 = \ker p_1$. This shows that $0 = \langle p_1 p_1^* v_2, v_2 \rangle = \langle p_1^* v_2, p_1^* v_2 \rangle$; hence $p_1^* v_2 = 0$, and $v_0 = 0$ in view of the injectivity of p_0. The second equality shows that $p_3^* v_3 \in \operatorname{im} p_2 = \ker p_3$. As before, $p_3^* v_3 = 0$. Hence $v_2 \in \ker p_2 = \operatorname{im} p_1$, say $v_2 = p_1 w_1$. Then $0 = p_1^* v_2 = p_1^* p_1 w_1$. Taking the scalar product with w_1, we see that $0 = p_1 w_1 = v_2$. Iteration gives the desired injectivity.

The exactness of the symbol complex implies that the dimensions of the sums of the even and odd spaces agree (!). Hence the symbol of $\mathcal{P}$ is a map between spaces of the same dimension and injectivity implies invertibility.

6.11. Lemma. *We use the notation of Definition 6.10.*

(a) *The adjoint $\mathcal{P}^*$ of $\mathcal{P}$ is given by*

$$\mathcal{P}^* = (P + P^*)|_{C^\infty(M, \mathscr{E}^{odd})} : C^\infty(M, \mathscr{E}^{odd}) \to C^\infty(M, \mathscr{E}^{even}).$$

(b) $\ker \mathcal{P} = \bigoplus_k \ker(P + P^*)|_{C^\infty(M, \mathscr{E}^{2k})} = \bigoplus_k \ker \Delta_{2k}$

(c) $\ker \mathcal{P}^* = \bigoplus_k \ker(P + P^*)|_{C^\infty(M, \mathscr{E}^{2k+1})} = \bigoplus_k \ker \Delta_{2k+1}$

Proof. (a) is clear.

(b) The first equality holds by definition. As for the second we see from Example 6.3 that for $v = (v_0, v_2, \ldots)$

$$(P + P^*)v = (P_1^* v_2 + P_0 v_0, P_3^* v_4 + P_2 v_2, \ldots).$$

In view of the fact that $\operatorname{im} P_k^* \perp \operatorname{im} P_{k-1}$ we find that $(P + P^*)v = 0$ if and only if

$$\begin{aligned} P_1^* v_2 = 0 = P_0 v_0, \\ P_3^* v_4 = 0 = P_2 v_2, \\ \ldots \end{aligned}$$

This in turn is equivalent to the fact that

$$v \in \ker P_{2k} \cap \ker P_{2k+1}^* \overset{6.7}{=} \ker \Delta_{2k}.$$

(c) Similarly. □

6.12. Theorem. *For an elliptic complex* 6.1(1) *we obtain*

$$\begin{aligned} \operatorname{ind} \mathcal{P} &= \dim \ker \mathcal{P} - \dim \ker \mathcal{P}^* \\ &= \sum_{k=0}^{m} (-1)^k \dim \ker \Delta_k = \sum_{k=0}^{m} (-1)^k \dim H^k. \end{aligned} \tag{1}$$

Proof. The first equality is due to the fact that $\ker \mathcal{P}^* = (\operatorname{im} \mathcal{P})^\perp$ and that $\operatorname{im} \mathcal{P}$ is closed, since $\operatorname{im} P_k = \ker P_{k+1}$. The second equality is a consequence of Lemma 6.3 and the third follows from the Hodge Theorem 6.9. □

6.13. Remark. The number on the right hand side of 6.12(1) is called the Euler characteristic of the complex. For the deRham complex

$$0 \to \Omega^0 M \xrightarrow{d} \Omega^1 M \to \ldots \xrightarrow{d} \Omega^n M \to 0$$

the spaces $H^k(M)$ are called the de Rham cohomology classes and the number $\sum_{k=0}^{m}(-1)^k \dim H^k(M)$ is called the Euler characteristic of M.

6.14. Example. The theorem of Gauß-Bonnet asserts that for a compact, oriented Riemannian surface M one has

$$\operatorname{ind} d = \chi(M) = \frac{1}{2\pi}\int_M K dS,$$

where K is the Gauß curvature (the product of the two principal curvatures). This expresses the index of the complex in locally computable terms. The result is true for any metric!

In this 2-dimensional case, the Euler characteristic $\chi(M)$ is related to the genus g of M by the formula

$$\chi(M) = 2 - 2g.$$

6.15. More complexes.

(a) **Signature complex.** Let M be a compact oriented manifold of dimension $n = 4k$. We can then define a symmetric bilinear form on $H^{2k}(M)$ by

$$(\alpha, \beta) = \int_M \alpha \wedge \beta \overset{\dim = 4k}{=} \int_M \beta \wedge \alpha.$$

The signature of the associated quadratic form is called the *signature* of M, $\operatorname{sign}(M)$.

Moreover, the smooth scalar product on the fibers on T^*M (i.e. a Riemannian metric g on M) gives us an L^2-scalar product on all $\Omega^m(M)$, and we obtain an isomorphism $\star : \Omega^m(M) \to \Omega^{n-m}$ via

$$\int \star\omega \wedge \eta = \int \langle \omega, \eta \rangle \, d\mu_g, \quad \omega, \eta \in \Omega^m(M).$$

The map $\star$ satisfies $\star\star = \pm I$ with a suitable sign. Letting $\tau : i^{k+m(m-1)}\star$ we obtain an involution[13] on $\Omega^\bullet(M) = \bigoplus \Omega^m(M)$, so that $\Omega^\bullet(M) = \Omega^+(M) \oplus \Omega^-(M)$, where $\Omega^\pm(M)$ are the ± 1-eigenspaces for τ.

Since $d + \delta$ anti-commutes with τ, we may consider the operator $d + \delta : \Omega^+(M) \to \Omega^-(M)$. It turns out that

$$\operatorname{ind}(d + \delta) = \operatorname{sign} M.$$

Hirzebruch's signature theorem (1953) expresses the signature as an integral:

$$\operatorname{sign}(M) = (\pi i)^{-2k} \int_M L(M)$$

Here $L(M)$ is the L-genus of M, a $4k$-form made up from the Pontryagin classes of the tangent bundle (to be explained later).

[13] An involution is a map ι with $\iota^2 = I$. In particular, its eigenvalues are ± 1.

(b) **Dolbeault complex.** On a complex manifold M we have the $\bar{\partial}$-operator,

$$\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

The cohomology of the complex

$$0 \to \Omega^{0,0}(M) \xrightarrow{\bar{\partial}} \Omega^{0,1}(M) \xrightarrow{\bar{\partial}} \ldots \to 0$$

is called the Dolbeault cohomology of M. Here

$$\operatorname{ind}(\bar{\partial} + \bar{\partial}^*) = \sum (-1)^p \dim_{\mathbb{C}} H^p(M),$$

the so-called holomorphic Euler characteristic of M.

The setting generalizes to the case of a complex vector bundle E over M.

The Hirzebruch-Riemann-Roch theorem [10] of 1954 expresses the Euler characteristic $\chi(M, E)$ by an integral involving the Chern class of E and the Todd class of the tangent bundle of M. More on this in the next section.