5. Pseudodifferential Operators

Pseudodifferential operators are an indispensable tool in modern analysis. Understanding the basics of this theory is worthwhile in many respects.

Pseudodifferential operators originated from the study of singular integral equations. Probably the first paper, where a complete calculus was developed is Kohn and Nirenberg's article [13]. Good sources to read more are the books by HÃ¶rmander [11], Kumano-go [14], Shubin [20], and Taylor [22].

5.1. Symbols and operators. Let $P = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$ be a differential operator on $\mathbb{R}^n$ and $u \in \mathscr{S}$. The properties of the Fourier transform in Lemma 4.6 show that

$$\begin{aligned} Pu(x) &= \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u(x) = \sum_{|\alpha|\le m} a_\alpha(x)\mathscr{F}^{-1}(\xi^\alpha \hat{u}(\xi))(x) \\ &= \int e^{ix\xi} \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \hat{u}(\xi)\, đ\xi = \int e^{ix\xi} p(x,\xi)\hat{u}(\xi)\, đ\xi, \end{aligned}$$

where $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$. A first idea in the theory of pseudodifferential operators is to replace the polynomial p by more general functions, the so-called *symbols*, in order to obtain much larger classes of operators. There are many different classes of symbols. The most prominent ones are those introduced below.

In a next step one tries to understand operations with these operators like compositions, adjoints, and (approximate) inverses through associated operations on the level of symbols.

5.2. Definition. For $m \in \mathbb{R}$, $S^m = S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is the space of all functions $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α, β

$$\sup_{x,\xi} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \langle\xi\rangle^{|\alpha|-m} < \infty.$$

These suprema define a set of seminorms, which yield a FrÃ©chet topology for S^m. We call the elements of S^m symbols of order m. We let $S^{-\infty} := \bigcap_m S^m$ and call these symbols smoothing or regularizing.

Of course, if $m \in \mathbb{N}_0$ and $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ with $a_\alpha \in \mathscr{C}_b^\infty(\mathbb{R}^n)$[9], then $p \in S^m$.

5.3. Lemma. *For* $s \in \mathbb{R}$,

$$\partial_\xi^\alpha \langle\xi\rangle^s = O(\langle\xi\rangle^{s-|\alpha|}).$$

So $\langle\xi\rangle^s$ *is an* x*-independent symbol of order* s.

Proof. Since $\partial_{\xi_j}\langle\xi\rangle = \xi_j/\langle\xi\rangle$, we see that $\partial_\xi^\alpha\langle\xi\rangle^s$ is a linear combination of terms of the form $p(\xi)\langle\xi\rangle^{s-|\alpha|-l}$, where $l \le |\alpha|$ and p is a polynomial of degree l. □

The following is clear:

[9] This is the space of all smooth functions which are bounded together with all their derivatives.

5.4. Lemma. *If p and q are symbols of orders m_p and m_q, respectively, then pq is a symbol of order $m_p + m_q$, and $\partial_\xi^\alpha \partial_x^\beta p$ is a symbol of order $m_p - |\alpha|$.*

We call a function $h : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ positively homogeneous of degree m, if

$$h(x, \lambda\xi) = \lambda^m h(x, \xi) \text{ for all } \lambda > 0. \tag{1}$$

5.5. Lemma. *Let h be as above. Choose a function $\zeta : \mathbb{R}^n \to \mathbb{R}$ which vanishes near zero and is constant 1 outside a neighborhood of zero.*[10] *Then $p(\xi) = \zeta(\xi)h(\xi)$, which is understood to be zero near $\xi = 0$, is a symbol of order m.*

More generally, let $p_m(x, \xi) \in C^\infty(\mathbb{R}^n_\xi \setminus \{0\}, C_b^\infty(\mathbb{R}^n_x))$ (the subscripts denote with respect to which variable p_m has these properties), be positively homogeneous of degree m in ξ, and let ζ be as above. Then $\zeta p_m \in S^m$.

5.6. Definition. With a symbol p we associate an operator $\mathrm{op}(p)$, defined on $\mathscr{S}$, by

$$(\mathrm{op}(p)u)(x) = \int e^{ix\xi} p(x, \xi)\hat{u}(\xi)\, đ\xi.$$

We call $\mathrm{op}(p)$ the pseudodifferential operator with symbol p.

5.7. Remark. It is sometimes useful to consider also symbols $q = q(x, y, \xi)$, smooth on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and satisfying

$$\sup_{x,y,\xi} |\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma q(x, y, \xi)| \langle\xi\rangle^{|\alpha|-m} < \infty.$$

They are called double symbols or amplitudes. We associate with q the pseudodifferential operator $\mathrm{op}(q)$ on $\mathscr{S}$ by

$$\mathrm{op}(q)(u)(x) = \iint e^{i(x-y)\xi} q(x, y, \xi)u(y)\, đy đ\xi.$$

However, this does not yield more operators. In fact, it is possible to find a symbol $p = p(x, \xi) \in S^m$ with $\mathrm{op}(p) = \mathrm{op}(q)$, see Kumano-go [14, Chapter 2].

Continuity. Pseudodifferential operators are continuous on $\mathscr{S}(\mathbb{R}^n)$ and Sobolev spaces, as we shall see below. They are also continuous on $C_c^\infty(\mathbb{R}^n)$, Besov and Calderòn-Zygmund spaces. We will confine ourselves to the first two examples.

5.8. Lemma. *For $p \in S^m$, the operator $\mathrm{op}(p) : \mathscr{S} \to \mathscr{S}$ is continuous.*

Proof. Use differentiation under the integral and the identity

$$x^\alpha \,\mathrm{op}(p)u(x) = \int e^{ix\xi} D_\xi^\alpha (p(x, \xi)\hat{u}(\xi))\, đ\xi.$$

□

[10] Such functions are called zero-excision functions.

5.9. Example. $\mathrm{op}(\langle\xi\rangle^2) = \mathrm{op}(1+|\xi|^2) = I-\Delta$. Moreover, $\mathrm{op}(\langle\xi\rangle^m)\,\mathrm{op}(\langle\xi\rangle^\mu) = \mathrm{op}(\langle\xi\rangle^{m+\mu})$ for arbitrary m, μ.

The operator $\mathrm{op}(\langle\xi\rangle^m)$ defines an isometric isomorphism $H^s \to H^{s-m}$ for each s. Indeed, for $u \in H^s$ we have $\mathrm{op}(\langle\xi\rangle^m)u = \mathscr{F}^{-1}(\langle\xi\rangle^m\hat{u})$ and therefore

$$\begin{aligned}\|\,\mathrm{op}(\langle\xi\rangle^m)u\|^2_{H^{s-m}} &= \int |\mathscr{F}(\mathrm{op}(\langle\xi\rangle^m)u)|^2\langle\xi\rangle^{2(s-m)}\,d\xi\\ &= \int |\langle\xi\rangle^m\hat{u}(\xi)|^2\langle\xi\rangle^{2(s-m)}\,d\xi = \int |u(\xi)|^2\langle\xi\rangle^{2s}\,d\xi = \|u\|^2_{H^s}.\end{aligned}$$

This shows that $\mathrm{op}(\langle\xi\rangle^m)$ is an isometry. The fact that $\mathrm{op}(\langle\xi\rangle^{-m})\,\mathrm{op}(\langle\xi\rangle^m) = I$ implies the assertion.

5.10. Theorem. *If p is a symbol of order m, then*

$$\mathrm{op}(p) : H^s \to H^{s-m}$$

is continuous for each $s \in \mathbb{R}$. The operator norm can be estimated by finitely many of the symbol seminorms.

Many proofs of this result can be found. Using the isomorphisms in Example 5.9, the task is easily reduced to the case $m = s = 0$. A simple argument is given in [12]. Once the boundedness is established on standard Sobolev spaces, it follows easily on many weighted spaces, too, see e.g. [19].

5.11. Corollary. *If p is a regularizing symbol, then $\mathrm{op}(p)$ maps H^s to $\bigcap_t H^t \subseteq \mathscr{C}^\infty(\mathbb{R}^n)$ by Corollary 4.13.*

Asymptotic expansions. The calculus of pseudodifferential operators gains much power through another concept, namely that of asymptotic expansions.

5.12. Definition.

(a) Let $p \in S^m$ and $p_{m-j} \in S^{m-j}$. We say that p has the asymptotic expansion

$$p \sim \sum_{j=0}^{\infty} p_{m-j}, \tag{1}$$

provided that, for each N, $p - \sum_{j=0}^{N} p_{m-j} \in S^{m-N-1}$.

(b) A symbol $p \in S^m$ is *classical*, if there exists an expansion (1) with p_{m-j} satisfying $p_{m-j}(x, \lambda\xi) = \lambda^{m-j}p_{m-j}(x,\xi)$ for $|\xi|, \lambda \geq 1$. In this case we call p_m the *principal symbol* of p, provided p_m is non-zero on $\{|\xi| \geq 1\}$.

5.13. Remark. It is easy to see that an asymptotic expansion is unique modulo regularizing symbols. In particular, if all p_{m-j} can be taken to be zero, then p is regularizing.

5.14. Remark.

(a) Instead of asking homogeneity for $|\xi| \geq 1$ in Definition 5.12(b) we could have taken any other positive value, also depending on j.

(b) Sometimes the term 'principal symbol' is not used for p_m but for the homogeneous extension of p_m to $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. This avoids ambiguities of the definition of p_m for $|\xi| \leq 1$.

5.15. Theorem. *Let* $(p_{m-j})_{j=0}^{\infty}$ *be a sequence of symbols with* $p_{m-j} \in S^{m-j}$*. Then there exists a symbol* $p \in S^m$ *with* $p \sim \sum p_{m-j}$*.*

For the proof one chooses a null sequence (ε_j), $\varepsilon_j > 0$, and a zero-excision function ζ and lets $p = \sum_{j=0}^{\infty} \zeta(\varepsilon_j \xi) p_{m-j}(x, \xi)$. If the ε_j tend to zero sufficiently fast, the series converges and furnishes the desired element.

5.16. Lemma. $\langle \xi \rangle^m$ *is classical. Its principal symbol is* $|\xi|^m$ *for* $|\xi| \geq 1$*.*

Proof. For $|\xi| > 1$, $\langle \xi \rangle^m = |\xi|^m (1 + |\xi|^{-2})^{m/2} = |\xi|^m \sum_{j=0}^{\infty} \binom{m/2}{j} |\xi|^{-2j}$. □

5.17. Example. Let q and p be as in Remark 5.7. Then p can be shown to have the asymptotic expansion $p(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha q(x, y, \xi)|_{y=x}$.

According to 5.8 we can compose pseudodifferential operators as operators on $\mathscr{S}$. The following is a key result in the theory of pseudodifferential operators. A proof can be found in Kumano-go's book [14].

5.18. Theorem. *Let* p *and* q *be symbols of orders* m_p *and* m_q*, respectively. Then* $\mathrm{op}(p)\,\mathrm{op}(q) = \mathrm{op}(r)$ *for a symbol* r *of order* $m_p + m_q$*. It has the asymptotic expansion*

$$r(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha p(x, \xi) D_x^\alpha q(x, \xi).$$

Moreover, the formal L^2*-adjoint*[11] $\mathrm{op}(p)^*$ *of* $\mathrm{op}(p)$ *is of the form* $\mathrm{op}(p^*)$ *for a symbol* $p^* \in S^m$*. It has the expansion*

$$p^*(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{p(x, \xi)}.$$

If p *and* q *are classical, then so are* r *and* p^**, and their asymptotic expansions can be determined from those of* p *and* q *together with the asymptotic expansion formulae above.*

5.19. Corollary. *Let* $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$ *have disjoint support. Then* $\varphi\,\mathrm{op}(p)\psi$ *is a regularizing pseudodifferential operator. This follows from the fact that* $\varphi\,\mathrm{op}(p) = \mathrm{op}(\varphi p)$ *together with the expansion formula in Theorem 5.18 and Remark 5.13.*

Pseudodifferential operators on manifolds. For the rest of this section let M be a compact manifold of dimension n.

We assume M to be covered by a finite number of coordinate patches (M_j, κ_j), $j = 1, \ldots, J$, where $\kappa_j : T_j \subset \mathbb{R}^n \to M_j$ are homeomorphisms and the T_j are bounded. Moreover, we choose a partition of unity φ_j, $j = 1, \ldots, J$, subordinate to this cover[12], and functions $\psi_j \in C^\infty(M)$, $j = 1, \ldots, J$, with support in M_j and $\varphi_j \psi_j = \varphi_j$.

Given a function u on M we can write $u = \sum_{j=1}^{J} \varphi_j u$. Each of the elements $\varphi_j u$ is supported in a single coordinate neighborhood and therefore can be

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[11] The operator satisfying $\langle \mathrm{op}(p)^* u, v \rangle = \langle u, \mathrm{op}(p) v \rangle$ for $u, v \in \mathscr{S}$.

[12] This means that $\varphi_j \in C^\infty(M)$ and $\operatorname{supp} \varphi_j \subseteq M_j$.

pulled back to a C^∞-function on $\mathbb{R}^n$. The same is true for $\psi_j u$. An operator $P : C^\infty(M) \to C^\infty(M)$ can be written

$$P = \sum_{j=1}^{J} \varphi_j P \psi_j + \sum_{j=1} \varphi_j P(1 - \psi_j). \tag{1}$$

For $j = 1, \dots, J$, $\varphi_j P \psi_j$ operates in a single coordinate neighborhood. We can therefore transfer these operators to $\mathbb{R}^n$ via the coordinate map κ_j.

5.20. Definition. Let $P : C^\infty(M) \to C^\infty(M)$ be a linear operator written as in (1). We say that P is a pseudodifferential operator of order m, provided that

(i) For $j = 1, \dots, J$, the pullback of the operator $\varphi_j P \psi_j$ is a pseudodifferential operator on $\mathbb{R}^n$ with a symbol in S^m.

(ii) The operators $\varphi_j P(1 - \psi_j)$ are given by integral operators with C^∞-kernels, i.e. $\varphi_j P(1-\psi_j)u(x) = \int_M k(x,y)u(y)dS(y)$ with $k \in C^\infty(M \times M)$ and a smooth measure dS on M.

We call P classical, if the local symbols are classical.

5.21. Remark. A pseudodifferential operator P on the compact manifold M has order $-\infty$ (that is P has any order $m \in \mathbb{R}$), if and only if P is an integral operator with smooth kernel on M. Indeed, if each of the local symbols p_j is smoothing, then the identity (in local coordinates)

$$\int e^{ix\xi} p(x,\xi)\hat{u}(\xi) đ\xi = \int\int e^{i(x-y)\xi} p(x,\xi) u(y) đy đ\xi$$

implies that P can be written with the integral kernel

$$k(x,y) = \int e^{i(x-y)\xi} p(x,\xi) đ\xi.$$

Conversely, an analogous formula shows that an operator with smooth kernel can be written as a pseudodifferential operator with regularizing local symbols.

On $\mathbb{R}^n$ the description of the pseudodifferential operators with regularizing symbols is more difficult as one has to take into account the decay properties at infinity.

The following result is a consequence of the definition of pseudodifferential operators on manifolds together with Theorem 5.10 and the fact that integral operators with smooth kernels map each space $H^s(M)$ to $C^\infty(M) = \bigcap H^t(M)$.

5.22. Theorem. *Let $P : C^\infty(M) \to C^\infty(M)$ be a pseudodifferential operator of order m. Then $P : H^s(M) \to H^{s-m}(M)$ is continuous.*

Definition 5.20 associates with a pseudodifferential operator a collection of symbols, one for each coordinate neighborhood M_j.

Now we make the following observation: Let $\chi : V \to U$ be a diffeomorphism of open sets in $\mathbb{R}^n$. Then χ induces a pullback of operators: It takes an operator $Q : C_c^\infty(U) \to C^\infty(U)$ to $\chi^* Q : C_c^\infty(V) \to C^\infty(V)$ via

$$(\chi^* Q)u(x) = Q(u \circ \chi^{-1})(\chi(x)), \quad u \in C_c^\infty(V), x \in V.$$

Suppose additionally that Q is a pseudodifferential operator given by the symbol $q \in S^m$. Then χ^*Q can also be written as a pseudodifferential operator with a symbol $r \in S^m$. If q is classical, so is r, and its principal symbol r_m is given by

$$r_m(x,\xi) = q_m(\chi(x), (\partial\chi(x))^{-t}\xi), \tag{1}$$

where q_m is the principal symbol of q and $(\cdot)^{-t}$ is the inverse of the transpose. In other words:

5.23. Corollary. *The local principal symbols of a classical pseudodifferential operator P of order m on M transform like a function on the cotangent bundle of M (without the zero section) and thus define a map $\sigma(P)$ on $T^*M\backslash 0$, which is homogeneous of degree m, i.e. $\sigma(P)(x,\lambda\xi) = \lambda^m\sigma(P)(x,\xi)$, $x \in M, 0 \neq \xi \in T^*_xM$.*

5.24. Theorem. *Let P_1 and P_2 be two classical pseudodifferential operators of order m on M. Suppose that $\sigma(P_1) = \sigma(P_2)$. Then $P_1 - P_2$ is of order $m-1$ and hence*

$$P_1 - P_2 : H^s(M) \to H^{s-m}(M) \text{ is compact.}$$

Proof. The equality of the principal symbols implies that $P_1 - P_2$ has order $m-1$ in each local coordinate patch and thus globally. So it maps H^s to H^{s-m+1} which is compact in H^{s-m} by Theorem 4.18(c). □

5.25. Theorem. *Let P_1 and P_2 be classical pseudodifferential operators of orders m_1 and m_2 on M. Then the composition P_1P_2 is a pseudodifferential operator of order $m_1 + m_2$, and*

$$\sigma(P_1P_2) = \sigma(P_1)\sigma(P_2).$$

Similarly, the formal L^2-adjoint P_1^* of P_1 is again a classical pseudodifferential operator of order m_1 and

$$\sigma(P_1^*) = \sigma(P_1)^*,$$

the adjoint (i.e. here the complex conjugate) of the symbol of P_1.

Proof. This follows from Theorem 5.18. □

Vector bundles. In general the operators we are considering will not act on scalar functions but on sections of vector bundles over M. Here are a few basic facts, see [1] for more details.

A (complex) vector bundle over M consists of a topological space E, the *total space* of the vector bundle, a continuous map $\pi : E \to M$, the so-called projection, and a finite-dimensional complex vector space structure on each of the sets $E_x = \pi^{-1}\{x\}$, $x \in M$, which is compatible with the topology on E_x induced by E. The space E_x is called the *fiber* of E over x. Moreover one requires E to be *locally trivial*; this means that for every $x \in M$ there exists a neighborhood U of x in M and a homeomorphism

$$\varphi_U : E|_U := \pi^{-1}(U) \to U \times \mathbb{C}^k \tag{1}$$

for suitable k such that φ maps E_x to $\{x\} \times \mathbb{C}^k$ as a linear map. The dimension k then is necessarily locally constant. We shall additionally assume that it is constant on M and call it the dimension of E.

Usually, one does not specify the map π and the vector space structure on the fibers and simply speaks of the vector bundle E over M.

A continuous map $\varphi : E^1 \to E^2$ between two vector bundles E^1, E^2 with projections π_1 and π_2 over M is a (vector bundle) *morphism*, if it is linear on each fiber and satisfies $\pi_2\varphi = \pi_1$. It is called an isomorphism, if it is bijective and the inverse also is a vector bundle morphism. Isomorphic vector bundles will not be distinguished.

It is clear that vector bundles can be defined over arbitrary topological spaces. We shall consider here only smooth vector bundles over M in the sense that the transition maps $\varphi_{U_2}\varphi_{U_1}^{-1}|_{(U_1\cap U_2)\times\mathbb{C}^k}$ for two trivializations of E over subsets U_1 and U_2 with nonempty intersection in the sense of (1) are smooth.

A vector bundle is *trivial*, if it is isomorphic to the bundle $M\times\mathbb{C}^k$ with the product topology and the canonical projection $(x,z)\mapsto x$, $x\in M, z\in\mathbb{C}^k$. The trivial bundle of dimension k is often denoted by $\underline{k}$.

A *section* of E is a map $s : M \to E$ which associates to $x \in M$ an element in the fiber E_x. If E is trivial over $U \subset M$ as above, then there s can be identified with a map $U \to \mathbb{C}^k$. We shall speak of continuous, smooth or H^s sections depending on whether the map s has this property and write $\mathcal{C}(M,E), C^\infty(M,E)$ or $H^s(M,E)$ for the corresponding spaces of sections.

5.26. Example. (a) Apart from the trivial bundles, well-known vector bundles are the tangent bundle TM and the cotangent bundle T^*M. The associated projection maps associate to a vector or covector, respectively, its base point.

(b) A vector field on a smooth manifold is a section of the tangent bundle TM; a k-form (an element of $\Omega^k M$) is a smooth section of the bundle $\Lambda^k T^*M$ of antisymmetric k-covectors.

(c) Given a map $f : M_1 \to M_2$ and a vector bundle E over M_2 we obtain a vector bundle f^*E over M_1 (the *pull-back* of E via f) in the following way: The total space of f^*E is the subset of $M_1 \times E$ consisting of all pairs (x,e) such that $f(x) = \pi(e)$ with the projection $(x,e)\mapsto x$. Intuitively, we attach to x in M_1 the fiber $E_{f(x)}$ of E over $f(x)\in M_2$.

5.27. Definition. Let E^1, E^2 be smooth vector bundles over M of dimensions k_1 and k_2, respectively, and let $P : C^\infty(M,E^1) \to C^\infty(M,E^2)$ be a linear operator. Similarly as in Section 5 cover M by coordinate neighborhoods which are in addition so small that the vector bundles are trivial over them. Then decompose P as in (1). We call P a pseudodifferential operator of order m, if, in local coordinates, the operators $\varphi_j P\psi_j$ are given by $k_2 \times k_1$ matrices of pseudodifferential operators of order m, while the operators $\varphi_j P(1-\psi_j)$ have smooth integral kernels.

5.28. Example. The exterior derivative $d : \Omega^k M \to \Omega^{k+1}M$ is a differential operator acting on sections of vector bundles.

5.29. Remark. Let P be as in Definition 5.27. As in Corollary 5.23 we can associate with P the symbol $\sigma(P)$, which now is a morphism

$$\sigma(P) : \pi_0^*E^1 \to \pi_0^*E^2$$

between the pull-backs of the bundles E^1 and E^2 under the projection map $\pi_0 : T^*M \setminus 0 \to M$ of the cotangent bundle with the zero section removed. (We use the subscript 0 to distinguish π_0 from the projection $\pi : T^*M \to M$.) The results of Theorem 5.25 continue to hold. Note that in order to speak of the adjoint of P, the bundles E^1 and E^2 have to be endowed with a (smooth) hermitian structure. This can always be achieved by patching locally defined hermitian forms with the help of a partition of unity. Locally, the symbol of the adjoint then is given by the adjoint of the symbol matrix.

5.30. Theorem. *We use the notation introduced in Definition 5.27. Given a smooth morphism $p : \pi_0^*E^1 \to \pi_0^*E^2$ which is homogeneous of degree m in the fibers, there exists a classical pseudodifferential operator*

$$P : C^\infty(M, E^1) \to C^\infty(M, E^2)$$

with $\sigma(P) = p$. Moreover, we can choose the mapping $p \mapsto P$ in such a way that the operator norm of $P : H^s(M, E^1) \to H^{s-m}(M, E^2)$ depends continuously on p.

Proof. Choose a finite covering of M by coordinate patches M_j over which the bundles E^1 and E^2 are trivial and an associated partition of unity φ_j. Denote by $\tilde{p}_j$, $\tilde{\varphi}_j$ the functions p and φ_j in local coordinates over M_j. Choose $\theta_j \in C_c^\infty(\mathbb{R}^n)$, supported in the same coordinate patch with $\theta_j(x)\tilde{\varphi}_j(x) = \tilde{\varphi}_j(x)$. Moreover, choose a function η on $\mathbb{R}$ which is constant 1 outside a neighborhood of 0 and vanishes in a smaller neighborhood of 0. Then define a pseudodifferential operator $\tilde{P}_j$ of order m on $\mathbb{R}^n$ by

$$\tilde{P}_j = \operatorname{op}(\theta_j(x)\tilde{p}_j(x,\xi)\eta(|\xi|)\theta_j(y)).$$

It is classical, since p_j is homogeneous of degree m by assumption. It can be pulled back to M with the help of the coordinate maps. Call the resulting operator P_j. Finally let $P = \sum \varphi_j P_j$. By construction, $\sigma(P) = p$. □

5.31. Example. The symbol $\sigma(d)(x,\xi)$ of $d : \Omega^k M \to \Omega^{k+1} M$ is the map $\pi^*\Lambda^k T_x^* M \ni \eta \mapsto i\xi \wedge \eta \in \pi^*\Lambda^{k+1} T_x^* M$. In fact, for $\eta = \sum f_I dx^I \in \Omega^k M$ and $\xi = \sum \xi_j dx^j$ we have

$$d\eta = \sum_{j=0}^n \partial_{x_j} f_I(x) dx^j \wedge dx^I \overset{-i\partial_x = D_x}{=} \mathscr{F}^{-1} \sum_{j=0}^n i\xi_j \mathscr{F} f_I(\xi) dx^j \wedge dx^I = i\mathscr{F}^{-1}\xi \wedge \mathscr{F}\eta.$$

Ellipticity and parametrices. The decisive strength of pseudodifferential operators is the possibility to construct an approximate inverse up to smoothing terms, provided the principal symbol is invertible.

5.32. Definition. A pseudodifferential operator $P : C^\infty(M, E^1) \to C^\infty(M, E^2)$ is elliptic, if the symbol $\sigma(P) : \pi^*E^1 \to \pi^*E^2$ is an isomorphism, i.e. if $\sigma(P)(x,\xi)$ is invertible for each (x,ξ) in $T^*M \setminus 0$.

5.33. Theorem. *Let*

$$P : C^\infty(M; E^1) \to C^\infty(M, E^2) \tag{1}$$

be an elliptic classical pseudodifferential operator of order m. Then there exists a pseudodifferential operator Q of order $-m$ such that $PQ - I$ and $QP - I$ are pseudodifferential operators of order -1.

Proof. The inverse $q = \sigma(P)^{-1}$ is a smooth morphism $\pi_0^* E^2 \to \pi_0^* E^1$ which is homogeneous of degree $-m$. According to Theorem 5.30 we find a classical pseudodifferential operator Q of order $-m$ with $\sigma(Q) = q$. By Theorem 5.25

$$\sigma(PQ) = \sigma(P)\sigma(Q) = I \text{ and } \sigma(QP) = \sigma(Q)\sigma(P) = I.$$

Hence $PQ - I$ and $QP - I$ have order -1 by Theorem 5.24. □

5.34. Remark. Iterating the above process and using asymptotic summation for symbols, see Theorem 5.15, one can actually find a classical pseudodifferential operator $\tilde{Q}$ of order $-m$ such that $\tilde{Q}P - I$ and $P\tilde{Q} - I$ are regularizing operators.

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5.35. Theorem. *Let P be as in Theorem 5.33. Then*

$$P : H^s(M, E^1) \to H^{s-m}(M, E^2)$$

is a Fredholm operator for every $s \in \mathbb{R}$.

Proof. Choose Q as in Theorem 5.33. By Theorem 5.10 it defines a bounded operator $H^{s-m}(M, E^2) \to H^s(M, E^1)$. As $PQ - I$ and $QP - I$ have order -1, they are compact. Hence P is a Fredholm operator and Q is a Fredholm inverse to P. □

5.36. Theorem. (Elliptic regularity). *Given P as in Theorem 5.33 and f in $H^s(M, E^2)$, let u be a solution of the equation $Pu = f$ with u in some Sobolev space $H^{-N}(M, E^1)$, $N \in \mathbb{R}$. Then $u \in H^{s+m}(M, E^1)$.*

Proof. Let Q be as in Theorem 5.33 and $R = QP - I$. Then R is of order -1 and

$$H^{s+m}(M, E^1) \ni Qf = QPu = (I + R)u = u + Ru.$$

In particular, $u \in H^{s+m} + H^{-N+1}$. If $-N + 1 < s + m$ the argument can be iterated until we reach the assertion. □

5.37. Corollary. *If $Pu = 0$, then $u \in \cap H^s$. In other words: The kernel of an elliptic pseudodifferential operator consists of smooth sections.*

Another important fact:

5.38. Theorem. *Let E be a smooth vector bundle over M. For each choice of s and μ there exists a pseudodifferential operator Λ^μ of order μ such that $\Lambda^\mu : H^s(M, E) \to H^{s-\mu}(M, E)$ is an isomorphism.*

Proof. Let $c > 0$. In each local coordinate neighborhood choose the pseudodifferential operator with the symbol $\langle \xi, c\rangle^\mu I_E$, where $\langle \xi, c\rangle = (1 + |\xi|^2 + c^2)^{1/2}$. With the help of a partition of unity patch these operators to a pseudodifferential operator $L_c^{(\mu)}$ of order μ on M. Similarly, we obtain a pseudodifferential operator $L_c^{(-\mu)}$ of order $-\mu$ by starting from the local symbols $\langle \xi, c\rangle^{-\mu} I_E$. Using Theorem 5.18 one can show that $L_c^{(-\mu)} L_c^{(\mu)} = I + R_1(c)$ and $L_c^{(\mu)} L_c^{(-\mu)} = I + R_2(c)$, where the symbol seminorms for $R_1(c)$ and $R_2(c)$ tend to zero as $c \to \infty$. By taking c large, $L_c^{(\mu)}$ will be thus be invertible as an operator $H^s(M, E) \to H^{s-\mu}(M, E)$ and can be taken for Λ^μ. □

5.39. Remark. If Λ^μ is invertible as an operator $H^{s_0}(M,E) \to H^{s_0-\mu}(M,E)$ for some s_0, then it will be invertible for all $s \in \mathbb{R}$, see Beals [4].

5.40. Theorem. *Let $P : C^\infty(M,E^1) \to C^\infty(M,E^2)$ be a classical elliptic pseudodifferential operator of order m. Then the index of $P : H^s(M,E^1) \to H^{s-m}(M,E^2)$ is independent of s and so are the dimensions of the kernel and the complement of the image of P.*

Moreover, the index only depends on the stable homotopy class of the principal symbol within the class of invertible symbols.

5.41. Remark. We call two symbols $\sigma_1, \sigma_2 : \pi_0^* E^1 \to \pi_0^* E^2$ of pseudodifferential operators P_1 and P_2 of order m stably homotopic, if there exists a bundle F over M and a bundle homomorphism λ of $\pi_0^* F$, which is homogeneous of degree m in ξ and a multiple of the identity on S^*M, such that the maps $\sigma_1 \oplus \lambda, \sigma_2 \oplus \lambda : \pi_0^*(E^1 \oplus F) \to \pi_0^*(E^2 \oplus F)$ are homotopic. By the general theory of vector bundles it is no restriction to assume that F is trivial. So we can view $\sigma_1 \oplus \lambda$ as the symbol of $P_1 \oplus \Lambda$, where Λ is an invertible pseudodifferential operator of order m acting on sections of $\underline{k}$ over M with principal symbol λ. The same applies to P_2. Clearly, this change does not affect the index.

Proof of Theorem 5.40. Choose s_1 and s_2 in $\mathbb{R}$, and denote for the moment by P_{s_1} and P_{s_2} the extensions of P to H^{s_1} and H^{s_2} respectively. Let $\mu = s_2 - s_1$ and choose Λ^μ and $\Lambda^{-\mu}$ as in the proof of Theorem 5.38 with mutually inverse principal symbols. As both are isomorphisms, the operator $\Lambda^{-\mu} P_{s_1} \Lambda^\mu$ is an operator on H^{s_2} with the same index as P_{s_1} and the same principal symbol as P_{s_2}, namely $\sigma(P)$. By Theorem 5.24 both have the same index.

In view of the fact that the kernel is a subset of C^∞, its dimension is constant. Since the index is constant, also the dimension of the complement will be constant.

Suppose that $\tilde{P}$ is another classical elliptic pseudodifferential operator and that the symbols $\sigma(P)$ and $\sigma(\tilde{P})$ are homotopic after possibly adding a trivial bundle as explained in Remark 5.41. Let $\sigma_t, 0 \le t \le 1$, be a homotopy through invertible symbols with $\sigma_0 = \sigma(P)$, $\sigma_1 = \sigma(\tilde{P})$. According to Theorem 5.30 we find a continuous family of operators P_t with $\sigma(P_t) = \sigma_t$. In view of the stability of the index in Proposition 1.5, the indices of P_0 and P_1 coincide. Since P_0 and P_1 have the same principal symbols as P and $\tilde{P}$, respectively, we see that $\operatorname{ind} P = \operatorname{ind} \tilde{P}$. □

5.42. Remark. We have noticed that the kernel of P in Theorem 5.40 consists of smooth functions. In a similar way one can show that there is a space of sections in $C^\infty(M,E^2)$ which complements the range of P on any space $H^s(M,E^2)$, $s \in \mathbb{R}$. Hence the index of P on any of the Sobolev spaces coincides with that as an operator $C^\infty(M,E^1) \to C^\infty(M;E^2)$.

Another observation which is often useful:

5.43. Remark. Order reducing operators allow us to focus on the index theory of elliptic pseudodifferential operators of order zero. This has the additional advantage that the principal symbol is homogenous of degree zero and therefore is in a natural way a function on the cosphere bundle S^*M.