4. Fourier Transform and Sobolev Spaces

4.1. Rapidly decreasing functions. The space $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing functions on $\mathbb{R}^n$ is defined as the space of all $u \in C^\infty(\mathbb{R}^n, \mathbb{C})$ such that, for each choice of multi-indices[7] α, β

(1) $$q_{\alpha,\beta}(u) := \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| < \infty.$$

The seminorms $q_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{N}_0^n$, define a Fréchet (i.e. complete metric, translation invariant) topology for $\mathscr{S}$.

Equivalently, one can ask that, for all multi-indices α, β,

(2) $$\tilde{q}_{\alpha,\beta}(u) := \|x^\alpha \partial_x^\beta u\|_{L^2(\mathbb{R}^n)} < \infty.$$

A linear map $T : \mathscr{S} \to \mathbb{C}$ is continuous, provided that there exist finitely many seminorms q_{α_j,β_j}, $j = 1, \ldots, N$ such that

$$|Tu| \le C \max_j q_{\alpha_j,\beta_j}(u), \quad u \in \mathscr{S},$$

correspondingly for the $\tilde{q}$ seminorms.

What does it mean for $S : \mathscr{S} \to \mathscr{S}$ to be continuous?

4.2. Tempered distributions. The dual space of $\mathscr{S}$, i.e. the space of all continuous linear maps $\mathscr{S} \to \mathbb{C}$, is denoted by $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$ and called the space of tempered distributions.

4.3. Example.

(a) An example for an element in $\mathscr{S}'$ is the delta distribution δ given by

$$\delta u = u(0), \quad u \in \mathscr{S}.$$

The delta distribution is continuous, since $|\delta u| \le \sup |u(x)|$.

(b) We can also view certain functions as tempered distributions. For example, let f be a function which is measurable and of at most polynomial growth, i.e. $|f(x)| \le C(1 + |x|)^m$ for some $C > 0$ and $m \in \mathbb{N}$. Define $T_f : \mathscr{S} \to \mathbb{C}$ by

$$T_f\varphi = \int f(x)\varphi(x)\,dx, \quad \varphi \in \mathscr{S}$$

(the integral exists due to the rapid decay of φ). Then T_f is a tempered distribution.

Given two such functions f and g we have $T_f = T_g$ if and only if $f = g$ a.e.. In view of this fact one often identifies the tempered distribution T_f with the function f. One calls tempered distributions of the form T_f, for a suitable measurable function f, *regular.*

4.4. Operations on $\mathscr{S}'$. We can define the derivative $\partial_{x_j} U$ of a tempered distribution U by asking that

$$(\partial_{x_j} U)(\varphi) = U(-\partial_{x_j}\varphi), \quad \varphi \in \mathscr{S}.$$

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[7]We are using multi-index notation: For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ we let $|\alpha| = \alpha_1 + \ldots \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, and for $\xi \in \mathbb{R}^n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. Moreover, $D_x^\alpha u = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n} u$ and $D_{x_j} = -i\partial_{x_j}$.

Then $\partial_{x_j} U$ is again a tempered distribution; so we can also take derivatives of arbitrary order. Consider the case, where $U = T_f$ for a function $f \in C^1(\mathbb{R}^n)$ with both f and ∇f of moderate growth. Then, for $\varphi \in \mathscr{S}$,

$$(\partial_{x_j} T_f)(\varphi) = \int f(x)(-\partial_{x_j}\varphi)(x)\,dx = \int \partial_{x_j} f(x)\varphi(x)\,dx = T_{\partial_{x_j} f}(\varphi).$$

We can also multiply by a smooth function g, which is of at most polynomial growth in all derivatives, by letting

$$(gU)(\varphi) = U(g\varphi), \quad \varphi \in \mathscr{S}.$$

In particular we can take for g a polynomial. Note that

$$(gT_f)(\varphi) = \int gf\varphi\,dx = T_{fg}(\varphi),$$

i.e., $gT_f = T_{gf}$.

4.5. The Fourier transform. The Fourier transform is one of the most important tools in the theory of partial differential equations, due to the fact that it converts differentiation to multiplication and vice versa, see Lemma 4.6, below.

The Fourier transform of $u \in \mathscr{S}$ is the function $\mathscr{F}u$ or $\hat{u}$ on $\mathbb{R}^n$ defined by

$$\mathscr{F}u(\xi) = \hat{u}(\xi) = \int e^{ix\xi} u(x) đx$$

with $đx = (2\pi)^{-n/2} dx$.

Actually, the integral above makes sense for $u \in L^1(\mathbb{R}^n)$. An application of the dominated convergence theorem shows that $\mathscr{F}u$ is continuous. In fact, $\mathscr{F}u(\xi) \to 0$ as $|\xi| \to \infty$, i.e. $\mathscr{F}u \in C_0(\mathbb{R}^n)$ (Riemann-Lebesgue).

A few easy-to-see facts about the Fourier transform:

4.6. Lemma. *Let $u \in \mathscr{S}$.*

(a) $\mathscr{F}(D^\alpha u)(\xi) = \xi^\alpha (\mathscr{F}u)(\xi)$.

(b) $\mathscr{F}(x^\alpha u)(\xi) = (-D_\xi^\alpha)\mathscr{F}u(\xi)$.

(c) $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ *is continuous.*

As a consequence of 4.6(c), we can extend $\mathscr{F}$ to a continuous map, also denoted $\mathscr{F}$, from $\mathscr{S}'$ to $\mathscr{S}'$: We let

$$(\mathscr{F}U)(\varphi) = U(\mathscr{F}\varphi) \text{ for } U \in \mathscr{S}' \text{ and } \varphi \in \mathscr{S}.$$

A more difficult result is the following theorem. See [17, Section 7] for proofs.

4.7. Theorem. *For $u \in \mathscr{S}$, we have $\mathscr{F}^2(u)(x) = u(-x)$, i.e applying $\mathscr{F}$ twice results in a reflection of the function. Moreover,*

$$\int \mathscr{F}u(\xi)\overline{\mathscr{F}v(\xi)}d\xi = \int u(x)\overline{v(x)}dx, \tag{1}$$

i.e. $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ is an isometry with respect to the $L^2(\mathbb{R}^n)$ scalar product.

As a consequence of Theorem 4.7, we have $\mathscr{F}^4 = I$ and thus $\mathscr{F}^{-1}u = \mathscr{F}^3 u = \mathscr{F}u(-\cdot)$ on $\mathscr{S}$. Hence

$$(\mathscr{F}^{-1}u)(x) = \int e^{ix\xi} u(\xi)\, đ\xi, \quad u \in \mathscr{S}. \tag{2}$$

Moreover (1) implies:

4.8. Plancherel's theorem. *The Fourier transform extends to an isometric isomorphism of $L^2(\mathbb{R}^n)$.*

Proof. Each function $u \in L^2(\mathbb{R}^n)$ can be approximated by a sequence (u_k) of functions in $\mathscr{S}$ (even by functions in $C_c^\infty(\mathbb{R}^n)$). Theorem 4.7 implies that $(\mathscr{F}u_k)$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$ and thus has a limit in $L^2(\mathbb{R}^n)$. One lets $\mathscr{F}u = \lim \mathscr{F}u_k$. □

4.9. Japanese brackets and Peetre's inequality. It will be convenient to work with the function

$$\mathbb{R}^n \ni \xi \mapsto \langle\xi\rangle = (1 + |\xi|^2)^{1/2}. \tag{1}$$

It is smooth, strictly positive and grows like $|\xi|$ as $\xi \to \infty$. We have the basic ('Peetre's') inequality

$$\langle\xi + \eta\rangle^s \leq 2^{|s|/2} \langle\xi\rangle^s \langle\eta\rangle^{|s|}, \quad \xi, \eta \in \mathbb{R}^n, s \in \mathbb{R}.$$

Note that, for $s \in \mathbb{N}_0$, $\langle\xi\rangle^{2s}$ is polynomial in $\xi_1^2, \ldots, \xi_n^2$.

Peetre's inequality holds for $s = 2$, since

$$\begin{aligned} 1 + |\xi + \eta|^2 &\leq 1 + (|\xi| + |\eta|)^2 = 1 + |\xi|^2 + 2|\xi||\eta| + |\eta|^2 \\ &\leq 1 + 2(|\xi|^2 + |\eta|^2) \leq 2(1 + |\xi|^2)(1 + |\eta|^2) \end{aligned}$$

This implies the assertion for $s \geq 0$. For $s < 0$ we note that

$$\langle\xi\rangle^{-s} = \langle(\xi + \eta) - \eta\rangle^{-s} \leq 2^{-s/2} \langle\xi + \eta\rangle^{-s} \langle\eta\rangle^{-s}.$$

4.10. Sobolev spaces. These are the function spaces on which (pseudo-) differential operators act in a natural way.

For $s \in \mathbb{R}$ the Sobolev space of order s on $\mathbb{R}^n$ is the set $H^s = H^s(\mathbb{R}^n)$ of all tempered distributions U whose Fourier transform $\mathscr{F}U$ is regular and satisfies

$$\|U\|_{H^s}^2 = \int |\mathscr{F}U(\xi)|^2 \langle\xi\rangle^{2s}\, đ\xi < \infty.$$

Clearly, $H^0(\mathbb{R}^n) \cong L^2(\mathbb{R}^n)$ through the Fourier transform. For $s > 0$, H^s is a subset of L^2. In fact, we can describe H^s explicitly for nonnegative integers s.

4.11. Theorem. *For $s = k \in \mathbb{N}_0$ we have*

$$H^k = \{u \in L^2 : \partial_x^\alpha u \in L^2 \text{ whenever } |\alpha| \leq k\}.$$

Proof. '$\subseteq$'. If u in H^k, then $\partial_x^\alpha u$ in $H^{k-|\alpha|}$, since $\mathscr{F}(\partial_x^\alpha u)(\xi) = \xi^\alpha (\mathscr{F}u)(\xi)$ and $|\xi|^{2|\alpha|} \langle\xi\rangle^{2k-2|\alpha|} \leq \langle\xi\rangle^{2k}$.

'$\supseteq$' If $\partial_x^\alpha u$ in L^2, then also $\xi^\alpha \mathscr{F}u \in L^2$ for $|\alpha| \leq k$. Since $\langle\xi\rangle^{2k} = \sum_{|\alpha| \leq k} c_\alpha \xi^{2\alpha}$ for suitable c_α (multiply out), we obtain the assertion. □

4.12. Theorem. (Sobolev embedding theorem) *For $s > n/2$ the space $H^s(\mathbb{R}^n)$ embeds into $C_0(\mathbb{R}^n)$.*[8]

Proof. Let $U \in H^s$. Then $\mathscr{F}U \in L^1(\mathbb{R}^n)$, since

$$\begin{aligned}\int |\mathscr{F}U(\xi)|\,d\xi &= \int |\mathscr{F}U(\xi)|\langle\xi\rangle^s\langle\xi\rangle^{-s}\,d\xi\\ &\le \left(\int |\mathscr{F}U(\xi)|^2\langle\xi\rangle^{2s}\,d\xi\right)^{1/2}\left(\int \langle\xi\rangle^{-2s}\,d\xi\right)^{1/2} = C\,\|U\|_{H^s}\end{aligned}$$

where the constant C is the value of the square root of the second integral on the right hand side. This integral is finite, since $2s > n$.

As pointed out in 4.5, $\mathscr{F}^2U$ is a continuous function. On the other hand, we know from Theorem 4.7 combined with Theorem 4.8 that $\mathscr{F}^2U = U(-\cdot)$, so that also U is continuous. □

4.13. Corollary. *Iterating the argument in Theorem 4.11 we see that H^{s+k} consists of C_0^k-functions whenever $k \in \mathbb{N}_0$ and $s > n/2$.*

4.14. Weighted Sobolev spaces. For $\mathbf{s} = (s_1, s_2) \in \mathbb{R}^2$ we can define the weighted Sobolev space

$$H^{\mathbf{s}}(\mathbb{R}^n) := H^{s_1,s_2}(\mathbb{R}^n) := \langle x\rangle^{-s_2}H^{s_1}(\mathbb{R}^n).$$

It is clear from Definition 4.1(2) that

$$\mathscr{S}(\mathbb{R}^n) = \bigcap_{\mathbf{s}\in\mathbb{R}^2} H^{\mathbf{s}}(\mathbb{R}^n).$$

(More precisely, the right hand side should be written as a projective limit.) In the exercises, we will see that the dual space of the standard Sobolev space $H^s(\mathbb{R}^n)$ can be identified with $H^{-s}(\mathbb{R}^n)$ via the L^2 scalar product. From this we see that the dual space to the weighted Sobolev space $H^{\mathbf{s}}(\mathbb{R}^n)$ can be identified with $H^{-\mathbf{s}}(\mathbb{R}^n)$. We can then show that

$$\mathscr{S}'(\mathbb{R}^n) = \bigcup_{\mathbf{s}\in\mathbb{R}^2} H^{\mathbf{s}}(\mathbb{R}^n), \tag{1}$$

where, more accurately, the right hand side is an inductive limit.

4.15. Theorem. (Structure of tempered distributions) *Given $U \in \mathscr{S}'$, we find finitely many bounded continuous functions $u_1, \ldots, u_N$ and multi-indices $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N$ such that*

$$U = \sum_{j=1}^{N} x^{\alpha_j}\partial_x^{\beta_j}u_j.$$

Proof. This is a consequence of 4.14(1). □

4.16. Schwartz' kernel theorem. Let $k \in \mathscr{S}'(\mathbb{R}^{n+m})$. Then k defines an operator

$$K_k : \mathscr{S}(\mathbb{R}^m) \to \mathscr{S}'(\mathbb{R}^n)$$

by

$$K_k(u)(\varphi) = k(\varphi \otimes u),$$

[8] As a function in H^s, U is only determined up to a set of measure zero. Hence the precise statement should be that there is a representative for U in $C_0(\mathbb{R}^n)$.

where $(\varphi \otimes u)(x, y) = \varphi(x)u(y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Conversely, every continuous linear operator

$$K : \mathscr{S}(\mathbb{R}^m) \to \mathscr{S}'(\mathbb{R}^n)$$

is of the form $K = K_k$ for some $k \in \mathscr{S}'(\mathbb{R}^{n+m})$, and the map $k \mapsto K_k$ is an isomorphism.

4.17. Remark. Sobolev spaces can also be defined on a compact manifold M. A distribution on M is a continuous linear map $U : C^\infty(M) \to \mathbb{C}$. We can multiply U by a function φ by letting $(\varphi U)(u) = U(\varphi u)$ for $\varphi, u \in C^\infty(M)$.

In order to define Sobolev spaces on M, choose a partition of unity $\{\varphi_j : j = 1, \ldots, J\}$, where each member φ_j is supported in a single coordinate neighborhood. Given a distribution U on M, we say that $U \in H^s(M)$, provided $\varphi_j U$, considered in local coordinates, is an element of $H^s(\mathbb{R}^n)$, $n = \dim M$.

4.18. Theorem. *Let M be an n-dimensional compact manifold. Then*

(a) $H^s(M) \hookrightarrow H^t(M)$ *for* $s \geq t$.

(b) $H^{s+k}(M) \hookrightarrow C^k(M)$ *for* $s > n/2, k \in \mathbb{N}_0$.

(c) *The inclusion* $H^s(M) \hookrightarrow H^t(M)$ *is compact for* $s > t$.

While (a) is trivial and (b) is immediate from Corollary 4.13, (c) is a deeper result known as Rellich's theorem. A proof can be found in [21, Chapter 4, Proposition 3.4].