## 3. Toeplitz Operators

**3.1. Banach algebras and $C^*$-algebras.** A Banach algebra is a complex Banach space $\mathscr{B}$ which is also an algebra with respect to a multiplication that additionally satisfies

$$\|ab\| \le \|a\|\|b\|, \quad a, b \in \mathscr{B}.$$

It is called a $C^*$-algebra if $\mathscr{B}$ also carries an involution $*$ satisfying

$$(a+b)^* = a^* + b^*, (ca)^* = \bar{c}a^*, (ab)^* = b^*a^*, \text{ and } \|a^*a\| = \|a\|^2$$

for $a, b \in \mathscr{B}, c \in \mathbb{C}$. The $C^*$-algebra $\mathscr{B}$ is *unital* if it contains a unit 1 such that $a1 = 1a = a$ for all $a \in \mathscr{B}$.

Clearly, every closed symmetric subalgebra of $\mathcal{L}(H)$, where $H$ is a complex Hilbert space, is a $C^*$-algebra. The $*$ here is the usual adjoint.

We see immediately that $\mathcal{L}(H)$ (unital), $\mathcal{K}(H)$ (non-unital, if $\dim H = \infty$) and the Calkin algebra $\mathcal{L}(H)/\mathcal{K}(H)$ are $C^*$-algebras. The Calkin algebra carries the involution $[A]^* = [A^*]$.

Another example is the algebra $C(X)$ of all continuous functions on a compact Hausdorff space $X$. The involution here is complex conjugation: $f^*(x) = \overline{f(x)} = \overline{f}(x)$.

**3.2. Theorem.** *Let $\mathscr{A}$ and $\mathscr{B}$ be $C^*$-algebras and let $\phi : \mathscr{A} \to \mathscr{B}$ be a $C^*$-algebra morphism. Then $\|\phi\| \le 1$.*

*If $\mathscr{A}$ and $\mathscr{B}$ are unital and $\phi$ is injective with $\phi(1_{\mathscr{A}}) = 1_{\mathscr{B}}$, then $\phi$ even is an isometry, i.e. $\|\phi(a)\| = \|a\|$ for all $a \in \mathscr{A}$.*

**3.3. The Hardy Space.** We denote by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle in $\mathbb{C}$. Each function $u \in L^2(\mathbb{T})$ has a Fourier series $u(\cdot) = \sum_{k\in\mathbb{Z}} a_k e^{ik\cdot}$. The map $u \mapsto (a_k)_{k\in\mathbb{Z}}$ is an isomorphism $L^2(\mathbb{T}) \to l^2(\mathbb{Z})$.

On $\mathbb{T}$, $z = e^{it}$, so that $e^{ikt} = z^k$, $k \in \mathbb{Z}$, and we can write the Fourier series also in the form

$$u = \sum_{k\in\mathbb{Z}} a_k z^k.$$

By $P : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ we denote the orthogonal projection[4]

$$u = \sum_{k\in\mathbb{Z}} a_k z^k \mapsto Pu = \sum_{k\ge 0} a_k z^k. \tag{1}$$

Orthogonality follows from the fact that for $u = \sum a_k z^k, v = \sum b_k z^k$ in $L^2$ we have $\langle u, v\rangle = \sum_{k\in\mathbb{Z}} a_k b_k$, so that

$$\langle Pu, v\rangle = \sum_{k\ge 0} a_k b_k = \langle u, Pv\rangle.$$

Then

$$\mathcal{H}^2 = \mathcal{H}^2(\mathbb{T}) = \{u \in L^2(\mathbb{T}) : u = \sum_{k\ge 0} a_k z^k\} = \operatorname{im} P = \ker(I - P)$$

is a closed subspace of $L^2(\mathbb{T})$, the *Hardy space* on $\mathbb{T}$.

From (1) we see immediately that $\mathcal{H}^2(\mathbb{T})$ consists of all $u \in L^2(\mathbb{T})$ which have a holomorphic extension to $\{z \in \mathbb{C} : |z| < 1\}$.

---

[4] A bounded linear operator $P$ on a Hilbert space is an orthogonal projection, if $P^2 = P = P^*$.

Similarly, we can define the spaces $\mathcal{H}^p(\mathbb{T})$, $1 \le p \le \infty$, consisting of those $u \in L^p(\mathbb{T})$, for which $u = \sum_{k\ge 0} a_k z^k$. We note that for $f \in \mathcal{H}^\infty$ we have $fu \in \mathcal{H}^2$ all $\mathcal{H}^2$: In fact, the product is in $L^2(\mathbb{T})$, since $f \in L^\infty$, and it has no negative Fourier coefficients by multiplying the Fourier series.

**3.4. Toeplitz operators.** Let $f \in L^\infty(\mathbb{T})$. Then we can define the Toeplitz operator $T_f$ associated with $f$ by

$$T_f : L^2(\mathbb{T}) \to L^2(\mathbb{T}), u \mapsto PM_fPu,$$

where $M_f$ is the operator of multiplication by $f$ It has the following properties

(a) The map $f \mapsto T_f$ is linear and continuous from $L^\infty$ to $\mathcal{L}(L^2(\mathbb{T}))$.

(b) $T_f^* = T_{\overline{f}}$

(c) $T_{fg} = T_fT_g$ for $f, g \in \mathcal{H}^\infty$

(d) We may also consider $T_f$ as an operator on $\mathcal{H}^2(\mathbb{T})$; the above properties continue to hold.

*Proof.* (a) The map is clearly linear. Continuity follows from the fact that

$$\|P(fPu)\|_2 \le \|fPu\|_2 \le \|f\|_\infty \|Pu\|_2 \le \|f\|_\infty \|u\|_2,$$

where we have used that an orthogonal projection has norm $\le 1$.

(b) $\langle PfPu, v\rangle = \langle u, P\overline{f}Pv\rangle$.

(c) In view of the fact that $M_g$ maps $\mathcal{H}^2$ to $\mathcal{H}^2$, we have $T_{fg}u = P(fgPu) = PfPgPu \overset{P^2=P}{=} PfPPgPu = T_fT_gu$. □

**3.5. Lemma.** *Let $f \in C(\mathbb{T})$. Then $(I-P)M_fP \in \mathcal{K}(L^2(\mathbb{T}))$.*

*Proof.* By Weierstraß's theorem, each function in $C(\mathbb{T})$ can be approximated uniformly by trigonometric polynomials[5]. Hence there exists a sequence $(f_n)$ of trigonometric polynomials with $f_n \to f$ in $L^\infty(\mathbb{T})$. Then

$$\|(I-P)M_fP - (I-P)M_{f_n}P\| = \|(I-P)M_{f-f_n}P\| \le \|f - f_n\|_\infty \to 0.$$

Since the compact operators form a closed subset of the bounded operators, it suffices to prove the assertion for each $f_n$. By linearity, we may even restrict ourselves to the case where $f(z) = z^m$ for some $m \in \mathbb{Z}$. But then, for $u = \sum a_k z^k$, we have

$$(I-P)M_fPu = (I-P)\sum_{k\ge 0} a_k z^{k+m} = \sum_{k\ge 0, m+k<0} a_k z^k.$$

Hence $(I-P)M_fP$ is a finite rank operator and therefore compact. □

**3.6. Theorem.** *Let $f, g \in C(\mathbb{T})$. Then*

(a) *$T_fT_g - T_{fg}$ is a compact operator on $\mathcal{H}^2$*

(b) *$T_f \in \mathcal{K}(\mathcal{H}^2)$ if and only if $f = 0$.*

*Proof.* (a) $PM_fPM_gP = PM_fM_g - PM_f(I-P)M_gP = PM_{fg}P + K$, where $K$ is compact by Lemma 3.5.

[5] Trigonometric polynomials are functions of the form $\sum_{|k|\le N} a_k z^k$ for some $N$.

(b) Since $(I-P)M_fP$ is compact, we see that $PM_fP$ is compact if and only if $M_fP$ is compact. On $\mathcal{H}^2$, however, $M_fP = M_f$, and a multiplication operator $M_f$ is compact if and only if $f = 0$.[6] □

**3.7. The Toeplitz algebra.** We define the Toeplitz algebra $\mathscr{T}$ as the smallest closed subalgebra of $\mathcal{L}(\mathcal{H}^2)$ which contains all operators $T_f$, $f \in C(\mathbb{T})$. Since $T_f^* = T_{\overline{f}}$, this subalgebra is automatically closed under taking adjoints. Hence $\mathscr{T}$ is a $C^*$-algebra. Actually, the Toeplitz algebra is one of the most important examples of a $C^*$-algebra.

We have

(a) $\mathcal{K}(\mathcal{H}^2) \subseteq \mathscr{T}$.

(b) $\mathcal{K}(\mathcal{H}^2)$ is the closure of the ideal generated by all commutators $[T_f, T_g]$, $f, g \in C(\mathbb{T})$.

(c) $\mathscr{T}/\mathcal{K}(\mathcal{H}^2)$ is a $C^*$-algebra isomorphic to $C(\mathbb{T})$ via the $C^*$-algebra isomorphism $\phi : f \mapsto [T_f]$.

(d) Every element of $\mathscr{T}$ can be written (uniquely) in the form $T_f + K$ for some $f \in C(\mathbb{T})$ and $K \in \mathcal{K}(\mathcal{H}^2)$.

*Proof.* (a) It is sufficient to show that $\mathscr{T}$ contains all finite rank operators. Those are of the form

$$u \mapsto \langle u, z^m\rangle z^n \text{ for suitable } m, n \in \mathbb{N}_0,$$

i.e. the maps $u = \sum c_k z^k \mapsto c_m z^n = (c_m z^m)T_{z^{n-m}}$. For $m \in \mathbb{N}$

$$(T_1 - T_{z^m}T_{z^{-m}})u \;=\; \sum_{k\geq 0} c_k z^k - T_{z^m}\sum_{k\geq m} c_k z^{k-m} = \sum_{k=0}^{m-1} c_k z^k.$$

Using the above construction for $m+1$ and $m$, and taking the difference, we see that $\mathscr{T}$ contains all projections $\sum c_k z^k \mapsto c_m z^m$, $m \geq 0$. So I$\mathscr{T}$ contains the finite rank operators and therefore $\mathcal{K}(\mathcal{H}^2)$.

(b) Let us denote for the moment by $\mathcal{C}$ the closed ideal generated by all commutators. Every commutator is a compact operator by Theorem 3.6(a), so $\mathcal{C} \subseteq \mathcal{K}(\mathcal{H}^2)$. By Theorem 3.6(a) and the computation above,

$$T_{z^{-m}}T_{z^m} - T_{z^m}T_{z^{-m}} = T_1 - T_{z^m}T_{z^{-m}} \in \mathcal{C} \cap \mathcal{K}(\mathcal{H}^2)$$

for $m \in \mathbb{N}$. As we saw, these operators generate all finite rank operators. Hence $\mathcal{C} = \mathcal{K}(\mathcal{H}^2)$.

(c) $\mathscr{T}/\mathcal{K}(\mathcal{H}^2)$ carries the quotient norm and the involution $*$ defined by $[T]^* = [T^*]$ inherited from $\mathcal{L}(\mathcal{H}^2)$. In particular, it is $*$-invariant and a $C^*$-algebra. As we know, $C(\mathbb{T})$ also is a $C^*$-algebra. Let us check that the map $\phi : f \mapsto [T_f]$ is a $C^*$-algebra morphism:

(i) $\phi(f) + c\phi(g) = [T_f] + c[T_g] = [T_f + cT_g] = [T_{f+cg}] = \phi(f+cg)$, $f, g \in C(\mathbb{T})$, $c \in \mathbb{C}$.

(ii) $\phi(f)^* = [T_f]^* = [(T_f)^*] \overset{3.5}{=} [T_{\overline{f}}] = \phi(\overline{f})$

[6] Suppose that $f(z_0) \neq 0$. Then $f(z) \neq 0$ for all $z$ in a neighborhood $U$ of $z_0$. Choose a function $\phi \in C_c(U)$ which is equal to 1 near $z_0$. Then $M_{\phi f^{-1}}M_f = M_\phi$. An operator of this form cannot be compact, since, via a partition of unity, a finite sum of such operators furnishes the identity.

(iii) $\phi(f)\phi(g) = [T_f][T_g] \overset{3.6(a)}{=} [T_{fg}] = \phi(fg)$.
Moreover, we know from Theorem 3.6(b) that $\phi$ is injective. We will next show that it is also surjective. By definition, the operators of the form $T_f$ generate $\mathscr{T}$. Since $T_fT_g - T_{fg}$ is compact for all $f, g$, the elements of the form $[T_f]$, $f \in C(\mathbb{T})$, form a dense set in $\mathscr{T}/\mathcal{K}(\mathcal{H}^2)$. According to Theorem 3.2, $\phi$ is an isometry. Hence the image of $\phi$ is closed. Since it is also dense, it is all of $\mathscr{T}/\mathcal{K}(\mathcal{H}^2)$.

(d) This is an immediate consequence of the fact that $\phi$ is surjective. □

**3.8. Theorem.** *An operator $T_f$ is a Fredholm operator in $\mathcal{L}(\mathcal{H}^2)$ if and only if $f(z) \neq 0$ for all $z \in \mathbb{T}$. In this case, the index is given by*

$$\operatorname{ind} T_f = -\mathit{wind}\, f,$$

*where $\mathit{wind}\, f$ is the winding number of $f$ around the origin.*

Before going into the proof, we recall that for a piecewise smooth function $f$ the winding number is given by

$$\mathit{wind}\, f = \frac{1}{2\pi i}\int_f \frac{1}{z}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(e^{it})e^{it}}{f(e^{it})}\, dt.$$

If $f$ is merely continuous, we use the following lemma.

**3.9. Lemma.** *Given a continuous map $f : \mathbb{T} \to \mathbb{C}\setminus\{0\}$ there exists a unique $n \in \mathbb{Z}$ and a function $\psi \in C(\mathbb{T})$, such that $f(z) = z^n e^{\psi(z)}$. The number $n$ is the winding number of $f$.*

*Proof.* Suppose first that $|f(z) - 1| < 1$. Then we can write $f = e^\psi$ for $\psi = \ln f$. Suppose next that we have $f_1$ and $f_2$ with $|f_2(z) - f_1(z)| < \|f_1^{-1}\|_\infty^{-1}$. Then $|f_2(z)/f_1(z) - 1| < 1$ and, as above, we can write $f_2/f_1 = e^\psi$ or, equivalently $f_2 = f_1e^\psi$. Hence, if the assertion holds for $f_1$, then also for $f_2$. We know that we can approximate every continuous function uniformly by a trigonometric polynomial. So we can replace $f$ by a function of the form $\sum_{k=-N}^N a_kz^k$. Since we can write the latter in the form $z^{-N}\sum_{k=-N}^N a_kz^{k+N}$, we may even assume that $f = \sum_{k=0}^M b_kz^k$ for suitable $b_k$. This is a polynomial in $z$ which can be written as a product

$$\sum_{k=0}^M b_kz^k = c\prod_{k=1}^M (z - \lambda_k),$$

and it is clearly sufficient to show the assertion for each term. So let $f(z) = z - \lambda_k$. Since $f$ has no zero on $\mathbb{T}$, we have $|\lambda_k| \neq 1$. For $|\lambda_k| < 1$ we have

$$|f(z) - z| = |\lambda_k| < 1 = \|z^{-1}\|_\infty^{-1}$$

and hence $f(z) = ze^\psi$ for some $\psi$. For $|\lambda_k| > 1$ we have

$$\|(1 - \lambda_k^{-1}z) - 1\|_\infty < 1$$

and hence $1 - \lambda_k^{-1}z = e^\psi$ or $f = z - \lambda_k = \lambda_ke^\psi = e^{\tilde\psi}$ for suitable $\tilde\psi$. Summing up, we obtain the desired representation of the original $f$ in the form $f = z^ne^\psi$.

Suppose we could write $f = z^ne^\psi = z^me^\phi$ for $m, n \in \mathbb{Z}$ and $\phi, \psi \in C(\mathbb{T})$. Then $z^{m-n} = e^{\psi-\phi}$. This, however requires that $m = n$, since the function

$e^{\phi-\psi}$ can be connected by a homotopy through invertible functions to the constant function 1 (e.g. take $g_s = e^{s(\phi-\psi)}$, $0 \le s \le 1$), while this is not possible for $z^k$, $k \neq 0$, since these functions have winding number $k$. $\square$

We can now prove Theorem 3.8. Clearly, $T_f$ is a Fredholm operator in $\mathcal{L}(\mathcal{H}^2)$ if and only if $[T_f]$ is invertible in $\mathscr{T}/\mathcal{K}(\mathcal{H}^2)$. In view of the fact that $\mathscr{T}/\mathcal{K}(\mathcal{H}^2) \cong C(\mathbb{T})$, invertibility of $[T_f]$ requires the invertibility of $f$.

Supposing that $f$ is invertible, write $f(z) = z^n e^{\psi(z)}$ for some $n \in \mathbb{Z}$ and $\psi \in C(\mathbb{T})$. For $0 \le s \le 1$ consider the function $f_s(z) = z^n e^{s\psi(z)}$. Then $s \mapsto f_s$ is a continuous map from $[0,1]$ to non-vanishing functions on $\mathbb{T}$. Hence all operators $T_{f_s}$ are Fredholm operators, and their index is the same, so

$$\inf T_f = \operatorname{ind} T_{f_1} = \operatorname{ind} T_{f_0} = \operatorname{ind} T_{z^n} = -n,$$

where, for the last equality, we use the fact that $T_{z^n}$ acts like a shift operator on the functions in $\mathcal{H}^2$. $\square$