2. Fredholm Operators

In the sequel, let X and Y be Banach spaces over $\mathbb{R}$ or (better) $\mathbb{C}$.

2.1. Definition. An operator A in $\mathcal{L}(X, Y)$, the space of continuous linear operators from X to Y, is a Fredholm[1] operator, if

$$\dim \ker A < \infty \text{ and } \operatorname{codim} \operatorname{im} A := \dim(Y/\operatorname{im} A) < \infty.$$

In this case, one calls

$$\operatorname{ind} A = \dim \ker A - \operatorname{codim} \operatorname{im} A \in \mathbb{Z}$$

the index of A. The quotient $Y/\operatorname{im} A$ is called the cokernel of A.

Of course, the notion of Fredholm operator makes sense for arbitrary vector spaces X and Y, and many properties hold also in this general situation. The most useful results, however, namely Theorems 2.5 and 2.14, require Banach and Hilbert spaces, respectively.

The Fredholm property can be seen as a substitute for invertibility. If A is a Fredholm operator, the equation $Au = f$ has solutions whenever f belongs to a subspace of Y which has a finite-dimensional complement, while the space of solutions is at most finite-dimensional. Clearly, an invertible operator has index zero; not every operator of index zero, however, is invertible.

The notion of a Fredholm operator is only of interest for infinite-dimensional spaces X and Y. In fact, the isomorphism theorem tells us that $X/\ker A \cong \operatorname{im} A$. If both X and Y have finite dimension, then

$$\dim X - \dim \ker A = \dim \operatorname{im} A = \dim Y - \operatorname{codim} \operatorname{im} A$$

and therefore

$$\operatorname{ind} A = \dim X - \dim Y.$$

Hence, in this case, *every* operator A is a Fredholm operator and the index is actually independent of A.

2.2. Example. Let $X = Y = \ell^2(\mathbb{N})$ be the space of square integrable sequences

$$\ell^2(\mathbb{N}) = \{x = (x_1, x_2, \ldots) : x_j \in \mathbb{C}, \sum |x_j|^2 < \infty\}.$$

We define the operators S_L and S_R in $\mathcal{L}(\ell^2(\mathbb{N}))$, which shift the sequences to the left and the right, respectively, by

$$S_L(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots) \text{ and } S_R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Then S_L is surjective and has one-dimensional kernel so that $\operatorname{ind} S_L = 1$. The operator S_R is injective, and its range has codimension 1. Therefore $\operatorname{ind} S_R = -1$.

An important analytic fact:

2.3. Lemma. *A Fredholm operator has closed range.*

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[1] named after Erik Ivar Fredholm, 1866-1927. In fact, it would be more appropriate to name them (as it its common in the Russian literature) after Fritz Noether, 1884-1941.

Proof. Let A be a Fredholm operator and let $\pi : X \to \widetilde{X} := X/\ker A$ be the quotient map. Endow $\widetilde{X}$ with the quotient norm. Then A induces the injective mapping $A' : \widetilde{X} \to Y$ with range $\operatorname{im} A$. Next choose an algebraic complement Y_1 of $\operatorname{im} A$, and form the exterior sum $Z = \widetilde{X} \oplus Y_1$. Both $\widetilde{X}$ and Y_1 are closed subspaces of Z, since Z carries the topology of the exterior sum. Y_1 is finite-dimensional, and $\tilde{X}$ is a Banach space, so Z is a Banach space. The mapping $\widetilde{A} : Z \to Y$ defined by

$$\widetilde{A}(\widetilde{x}, y) = A'\widetilde{x} + y, \ \widetilde{x} \in \widetilde{X}, y \in Y_1,$$

is continuous and bijective. The inverse then is continuous by Theorem 1.3. This in turn shows that $\operatorname{im} A = \widetilde{A}(\widetilde{X}) = [\widetilde{A}^{-1}]^{-1}(\widetilde{X})$ is closed. □

The index behaves well under compositions:

2.4. Lemma. *Let X, Y, and Z be Banach spaces. If $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, Z)$ both are Fredholm operators, then so is the composition BA, and*

$$\operatorname{ind}(BA) = \operatorname{ind} B + \operatorname{ind} A.$$

Proof. The proof, below, is purely algebraic and works on general vector spaces. Choose $X_1 \le X$ with $X = \ker A \oplus X_1$ und $Y_1, Y_2, Y_3 \le Y$ such that

$$\begin{aligned} Y &= \operatorname{im} A + \ker B \oplus Y_3; \qquad (1)\\ \ker B &= (\operatorname{im} A \cap \ker B) \oplus Y_2;\\ \operatorname{im} A &= (\operatorname{im} A \cap \ker B) \oplus Y_1. \end{aligned}$$

Then

$$Y = Y_1 \oplus (\operatorname{im} A \cap \ker B) \oplus Y_2 \oplus Y_3.$$

Next choose $Z_1 \le Z$ with

$$Z = B(Y_1) \oplus B(Y_3) \oplus Z_1.$$

Note: For $x_1 \in Y_1$ and $x_3 \in Y_3$ with $Bx_1 = Bx_3$ we have $x_3 - x_1 \in \ker B$, thus $x_3 \in Y_1 + \ker B \subseteq \operatorname{im} A + \ker B$, hence $x_3 = 0$ by (1).

Moreover, $\dim(\operatorname{im} A \cap \ker B) = \dim\ \ker B - \dim\ Y_2$.

$B|_{Y_1 \oplus Y_3}$ is injective, and therefore

$$\dim\ B(Y_3) = \dim\ Y_3 = \dim\ Y_2 \oplus Y_3 - \dim\ Y_2 = \operatorname{codim}\ \operatorname{im} A - \dim\ Y_2.$$

Furthermore $\operatorname{im} BA = B(Y_1), \ker BA = \ker A \oplus \{x \in X_1 : Ax \in \ker B\}$. We conclude

$$\begin{aligned} \operatorname{ind} BA &= \dim\ \ker BA - \operatorname{codim}\ \operatorname{im} BA\\ &= \dim\ \ker A + \dim\ \operatorname{im} A \cap \ker B - (\dim\ B(Y_3) + \dim\ Z_1)\\ &= \dim\ \ker A + \dim\ \ker B - \dim\ Y_2 - \operatorname{codim}\ \operatorname{im} A + \dim\ Y_2\\ &\quad - \operatorname{codim}\ \operatorname{im} B\\ &= \operatorname{ind} B + \operatorname{ind} A. \end{aligned}$$

□

In connection with Example 2.2 we see that there exist Fredholm operators of every index on $\ell^2(\mathbb{N})$.

The following theorem underlines the point of view that Fredholm operators are 'almost' invertible. They have inverses up to finite rank operators.

What is remarkable is that it is sufficient to find inverses up to compact operators to guarantee the Fredholm property. This is Atkinson's theorem.

2.5. Theorem. *For $A \in \mathcal{L}(X,Y)$ the following are equivalent:*

(i) *A is a Fredholm operator.*

(ii) *There exist $L, R \in \mathcal{L}(Y,X)$ such that*

$$LA - I = F_1 \quad \text{and} \quad AR - I = F_2$$

are finite rank operators in the respective spaces.

(iii) *There exist $\tilde{L}, \tilde{R} \in \mathcal{L}(Y,X)$ such that*

$$\tilde{L}A - I = K_1 \quad \text{and} \quad A\tilde{R} - I = K_2$$

are compact operators in the respective spaces.

In case one of the conditions holds, we can choose $L = R$ and $\tilde{L} = \tilde{R}$, respectively.

We call L and $\tilde{L}$ Fredholm inverses to A. We then see that L and $\tilde{L}$ also are Fredholm operators. The same applies to R and R_1.

The equivalence of (i) and (ii) is a purely algebraic fact. Here is a proof:

(i)⇒(ii) Choose complements X_1 of $\ker A$ in X und Y_1 of $\operatorname{im} A$ in Y.

Then $A_1 = A|_{X_1} : X_1 \to \operatorname{im} A$ is an isomorphism. Denote by $P_{X_1}, P_{\ker A}$, $P_{Y_1}, P_{\operatorname{im} A}$ the associated projections onto the spaces in the subscript along the complementary space. For $L = A_1^{-1} P_{\operatorname{im} A} = R$ we obtain

$$\begin{aligned} LA &= P_{X_1} = I - P_{\ker A} \in I + \mathcal{F}(X), \text{ and} \\ AR &= P_{\operatorname{im} A} = I - P_{Y_1} \in I + \mathcal{F}(F), \end{aligned}$$

where $\mathcal{F}(\cdot)$ here denotes the finite rank operators. Note that $P_{\operatorname{im} A}$ is continuous, since $\operatorname{im} A$ is closed.

(ii)⇒(i) For $x \in \ker A \subseteq \ker LA$ we have $0 = LAx = x - F_1 x$, hence $\dim\ \ker A \le \dim\ \operatorname{im} F_1 < \infty$. Moreover,

$$Y = \operatorname{im} Y_2 + \operatorname{im}(I - F_2) = \operatorname{im} F_2 + \operatorname{im} AR.$$

As $\operatorname{im} AR \subseteq \operatorname{im} A$, the codimension of $\operatorname{im} A$ is finite.

This proves that (i) ⇔ (ii). Before proving that (i) ⇔ (iii) let us note a consequence:

2.6. Lemma. *Let $A \in \mathcal{L}(X,Y)$ be a Fredholm operator. Then $A + F$ is a Fredholm operator for each $F \in \mathcal{F}(X,Y))$ and $\operatorname{ind}(A + F) = \operatorname{ind} A$.*

We will see later that the statement also holds for F compact.

Proof. The equivalence of (i) and (ii) in Theorem 2.5 shows that $A + F$ is a Fredholm operator. In fact, if L is a Fredholm inverse for A, then also for $A + F$.

Now first let $X = Y$ and $A = I$. Decompose $X = X_1 \oplus (\ker F \cap \operatorname{im} F) \oplus X_2 \oplus X_3 = X_1 \oplus X_4$, where

$$\begin{aligned} X_1 \oplus (\ker F \cap \operatorname{im} F) &= \ker F; \\ (\ker F \cap \operatorname{im} F) \oplus X_2 &= \operatorname{im} F; \\ (\ker F \cap \operatorname{im} F) \oplus X_2 \oplus X_3 &=: X_4. \end{aligned}$$

Write $T = I + F$. We then have

$$TX_4 \stackrel{T=I+F}{=} X_4 + FX_4 \stackrel{\operatorname{im} F \le X_4}{\le} X_4.$$

On $X_2 \oplus X_3$, which is a complement to $\ker F$, F is injective, so $\dim X_2 \oplus X_3 = \dim \operatorname{im} F < \infty$. Hence also X_4 is finite dimensional and T-invariant. On X_1, $T = I$. On the finite dimensional space X_4 we have $\dim \ker T + \dim \operatorname{im} T = \dim X_4$, and so

$$\dim \ker T = \dim \ker T|_{X_4} = \operatorname{codim} \operatorname{im} T|_{X_4} = \operatorname{codim} \operatorname{im} T.$$

Next let $F \in \mathcal{F}(X,Y)$ and $A \in \mathcal{L}(X,Y)$ be an arbitrary Fredholm operator and choose L such that $LA = I + F_1$ with $F_1 \in \mathcal{F}(X)$. By what we have shown and Lemma 2.2,

$$\begin{aligned} 0 &= \operatorname{ind}(I + F_1) = \operatorname{ind} L + \operatorname{ind} A, \\ 0 &= \operatorname{ind}(I + F_1 + LF) = \operatorname{ind} L(A+F) = \operatorname{ind} L + \operatorname{ind}(A+F), \end{aligned}$$

hence $\operatorname{ind}(A+F) = \operatorname{ind} A = -\operatorname{ind} L$. □

We need further preparations for the proof of the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 2.5. It will be shown after Lemma 2.9.

2.7. Lemma. *If $K \in \mathcal{K}(X)$, then $I - K$ has finite-dimensional kernel and closed range.*

Proof. On $\ker(I-K)$, K coincides with the identity. Since K is compact, the unit ball of $\ker(I-K)$, i.e. $\{x \in X : \|x\| < 1, x = Kx\}$ is relatively compact, hence finite-dimensional by 1.6. Lemma 1.11 therefore implies that $\ker(I-K)$ is the image of a continuous projection, say P. Letting $X_1 = \ker P$, we have a closed subspace of X with $X = \ker(I-K) \oplus X_1$; the sum is topologically direct by Lemma 1.9. Write $T = I - K$. Let $y \in \overline{\operatorname{im} T}$ and let $(y_n)_n$ be a sequence in $\operatorname{im} T$ converging to y. Since $\operatorname{im} T = TX_1$, there is a sequence (x_n) in X_1 with $y_n = Tx_n = x_n - Kx_n$. In case (x_n) is bounded, the compactness of $K((x_n)_n)$ implies that we can choose a subsequence (x_{n_j}) such that (Kx_{n_j}) is convergent. Then $x_{n_j} = y_{n_j} + Kx_{n_j}$ is also convergent. Let x_0 be the limit. The continuity of T implies that

$$Tx_0 = \lim_{j\to\infty} Tx_{n_j} = \lim_{j\to\infty} y_{n_j} = y.$$

Hence, in this case, $y \in \operatorname{im} T$. We shall next see that the complementary case, where (x_n) is unbounded, is not possible: Assume $0 < \|x_n\| \to \infty$. Letting $z_n = x_n/\|x_n\|$ we obtain a bounded sequence, hence a subsequence (z_{nj}) with Kz_{n_j} converging to, say, x_0. Then $Tz_{n_j} = Tx_{n_j}/\|x_{n_j}\| = y_{n_j}/\|x_{n_j}\|$ is a null sequence. Moreover,

$$\lim_{j\to\infty} z_{n_j} = \lim_{j\to\infty}(Tz_{n_j} + Kz_{n_j}) = \lim_{j\to\infty} Kz_{n_j} = x_0.$$

Since all z_{n_j} are elements of X_1, so is x_0, for X_1 is closed. Now $Tx_0 = \lim Tz_{n_j} = 0$, so $x_0 \in \ker T \cap X_1 = \{0\}$. This is a contradiction, since $\|x_0\| = \lim_{j\to\infty} \|z_{n_j}\| = 1$. □

2.8. Riesz' Lemma. Let X be a normed space and $Y < X$ a closed subspace. Given $0 < \delta < 1$, there is an $x \in X \setminus Y$ with $\|x\| = 1$ and

$$\inf\{\|x - y\| : y \in Y\} \ge \delta.$$

Proof. Let $x_0 \in X \setminus Y$ and $d = \inf\{\|x_0 - y\| : y \in Y\}$. Since Y is closed, d is positive. Given an $\varepsilon > 0$ we can find a $y_\varepsilon \in Y$ with $d \le \|x_0 - y_\varepsilon\| \le d + \varepsilon$. Choose $\varepsilon > 0$ so small that $d/(d+\varepsilon) \ge \delta$, and let $x = (x_0 - y_\varepsilon)/\|x_0 - y_\varepsilon\|$. Then $x \in X \setminus Y$ and $\|x\| = 1$. Moreover, for every $z \in Y$ we have $y_\varepsilon + \|x_0 - y_\varepsilon\| z \in Y$ and therefore

$$\|x - z\| = \left\| \frac{x_0 - y_\varepsilon}{\|x_0 - y_\varepsilon\|} - z \right\| = \frac{\|x_0 - (y_\varepsilon + \|x_0 - y_\varepsilon\| z)\|}{\|x_0 - y_\varepsilon\|} \ge \frac{d}{d+\varepsilon} \ge \delta.$$

□

2.9. Lemma. *$I - K$ is a Fredholm operator for every $K \in \mathcal{K}(X)$.*

Proof. Let $T = I - K$. It has finite dimensional kernel and closed range by Lemma 2.7. If $\operatorname{im} T < X$, then apply Riesz' lemma to find a $y_1 \in X \setminus \operatorname{im} T$ with $\|y_1\| = 1$ and $\inf\{\|y - y_1\| : y \in \operatorname{im} T\} \ge 1/2$. Since $\operatorname{span}\{y_1\}$ is one-dimensional, $M_1 := \operatorname{im} T \oplus \operatorname{span}\{y_1\}$ is topologically direct and a closed subspace of X by Lemma 1.10. If $\operatorname{im} T$ had infinite codimension, we could find a sequence (y_n) with $\|y_n\| = 1$ and closed subspaces

$$M_n = \operatorname{im} T \oplus \operatorname{span}\{y_1\} \oplus \cdots \oplus \operatorname{span}\{y_n\}$$

satisfying $\inf\{\|y_n - y\| : y \in M_{n-1}\} \ge 1/2$. We then had, for $j > k$,

$$\|Ky_j - Ky_k\| = \|y_j - Ty_j - y_k + Ty_k\| \ge 1/2,$$

since $Ty_j + y_k - Ty_k \in M_k$. This is a contradiction to the compactness of K. □

We are now ready to show the remaining part of Theorem 2.5, namely that $A \in \mathcal{L}(X, Y)$ is a Fredholm operator if and only if it is invertible modulo compact operators. Since we know already that the Fredholm property is equivalent to being invertible modulo finite rank operators, we only have to show that invertibility modulo compacts implies the Fredholm property.

By Lemma 2.9, $I + K_1$ and $I + K_2$ in Theorem 2.5(iii) are Fredholm operators. Applying the equivalence already proven, we find $L_1 \in \mathcal{L}(X)$, $R_1 \in \mathcal{L}(Y)$ such that

$$\begin{aligned} L_1\tilde{L}A - I &= L_1(I + K_1) - I \in \mathcal{F}(X) \text{ and} \\ A\tilde{R}R_1 - I &= (I + K_2)R_1 - I \in \mathcal{F}(Y). \end{aligned}$$

Now the equivalence (i) $\Leftrightarrow$ (ii) implies that A is a Fredholm operator.

In order to see that we may choose the same operator for both $\tilde{L}$ and $\tilde{R}$, we check that $\tilde{L} - \tilde{R} \in \mathcal{K}(Y, X)$:

$$\tilde{L} = \tilde{L}I = \tilde{L}(A\tilde{R} - K_2) = (\tilde{L}A)\tilde{R} - \tilde{L}K_2 = (I + K_1)\tilde{R} - \tilde{L}K_2 \equiv \tilde{R}$$

modulo compact operators.

2.10. Remark. Let $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$ be the canonical epimorphism. Theorem 2.5 shows that $A \in \mathcal{L}(X)$ is a Fredholm operator if and only if $\pi(A)$ is invertible in the so-called Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$.

2.11. Theorem. *We denote by* $\mathrm{Fred}(X,Y)$ *the set of all Fredholm operators in* $\mathcal{L}(X,Y)$ *and by* $\mathrm{Fred}_n(X,Y)$ *those of index* n*. Then* $\mathrm{Fred}_n(X,Y)$ *is open in* $\mathcal{L}(X,Y)$*. In particular,*

$$\mathrm{ind} : \mathrm{Fred}(X,Y) \to \mathbb{Z}$$

is continuous with respect to the topology of $\mathcal{L}(X,Y)$ *and therefore constant on the connected components.*

Proof. Let $A \in \mathrm{Fred}_n(X,Y)$ and

$$d = \sup\left\{\|L\|^{-1} : LA - I \in \mathcal{F}(X), AL - I \in \mathcal{F}(Y)\right\} > 0.$$

We shall see that, for each $S \in \mathcal{L}(X,Y)$ with $\|S\| < d$, we have $A + S \in \mathrm{Fred}_n(X,Y)$: Choose $L \in \mathcal{L}(Y,X)$ such that $LA - I = F_1 \in \mathcal{F}(X)$ and $\|S\| < 1/\|L\|$. Since $\|LS\| < 1, I + LS$ is invertible by Theorem 1.1, so $L(A+S) = I + LS + F_1$ is Fredholm of index zero by Lemma 2.6. Hence we can find $L_1 \in \mathcal{L}(X)$ with

$$L_1L(A+S) - I \in \mathcal{F}(X).$$

By the same argument, $(A+S)L$ is a Fredholm operator, so we find $R_1 \in \mathcal{L}(Y)$ with

$$(A+S)LR_1 - I \in \mathcal{F}(Y).$$

This shows that $A+S$ is a Fredholm operator. We also have

$$\begin{aligned} 0 &= \mathrm{ind}\, LA = \mathrm{ind}\, L + \mathrm{ind}\, A, \text{ and} \\ 0 &= \mathrm{ind}\, L(A+S) = \mathrm{ind}\, L + \mathrm{ind}(A+S), \end{aligned}$$

so $\mathrm{ind}\, A = \mathrm{ind}\, A + S$. ◻

2.12. Corollary. $\mathrm{ind}(I+K) = 0$ *whenever* $K \in \mathcal{K}(X)$*.*

Proof. The mapping $\alpha : [0,1] \to \mathrm{Fred}(X)$ given by $\alpha(s) = I + sK$ is continuous in the topology of $\mathcal{L}(X)$. The index therefore is constant. Since it is zero for $s = 0$, we get the assertion. ◻

2.13. Lemma. *If* A *in* $\mathcal{L}(X,Y)$ *is a Fredholm operator and* $K \in \mathcal{K}(X,Y)$*, then* $A+K$ *is a Fredholm operator, and* $\mathrm{ind}(A+K) = \mathrm{ind}\, A$*.*

Proof. Let L be a Fredholm inverse for A modulo compact operators. Then L is also a Fredholm inverse for $A+K$. Hence $A+K$ is a Fredholm operator. By Corollary 2.12

$$\mathrm{ind}\, LA = 0 = \mathrm{ind}\, L(A+K)$$

and $\mathrm{ind}\, A = -\,\mathrm{ind}\, L = \mathrm{ind}(A+K)$. ◻

There exists a considerable extension of Lemma 2.13 due to H¨ormander, see [11, Theorem 19.1.10] for details.

2.14. Theorem. *Let* $\mathcal{T}$ *be a compact space and* $A_t \in \mathcal{L}(X,Y)$*,* $B_t \in \mathcal{L}(Y,X)$*,* $t \in \mathcal{T}$*, be strongly continuous as functions of* t.[2] *Suppose that the operators*

$$K_{1,t} = A_tB_t - I \text{ and } K_{2,t} = B_tA_t - I$$

are uniformly compact in the sense that the sets

$$M_1 = \{K_{1t}u : u \in Y, \|u\| \le 1, t \in \mathcal{T}\} \text{ and } M_2 = \{K_{2t}u : u \in X, \|u\| \le 1, t \in \mathcal{T}\}$$

[2] i.e. for $x \in X$, the maps $t \mapsto A_tu$ and $t \mapsto B_tu$ are continuous

have compact closure in Y and X, respectively.

Then A_t and B_t are Fredholm operators, $\dim\ker A_t$ *and* $\dim\ker B_t$ *are upper semicontinuous,*[3] *and* $\operatorname{ind} A_t = -\operatorname{ind} B_t$ *are locally constant on $\mathcal{T}$. In particular, they are constant, if $\mathcal{T}$ is connected.*

If we consider Hilbert spaces, then we get more results:

2.15. Lemma. *Let H be an infinite-dimensional Hilbert space. Then*

$$\operatorname{ind} : \operatorname{Fred}(H) \to \mathbb{Z}$$

is surjective.

Proof. We have $H \simeq l^2(\mathbb{N}) \oplus l^2(\Gamma)$, where Γ is a suitable index set. Similarly as in Example 2.2 define the right and left shift operators S_R and S_L, respectively, by

$$\begin{aligned} S_R((x_1, x_2, \ldots) \oplus y) &= ((0, x_1, x_2, \ldots) \oplus y) \\ S_L((x_1, x_2, \ldots) \oplus y) &= ((x_2, x_3, \ldots) \oplus y) \end{aligned}$$

for $x = (x_1, x_2, \ldots) \in l^2(\mathbb{N})$ and $y \in l^2(\Gamma)$. Then $\operatorname{ind} S_R = -1$ and $\operatorname{ind} S_L = 1$. □

According to Theorem 2.11 the index is constant on the connected components of $\operatorname{Fred}(X)$. In the Hilbert space situation, the index even classifies the components of the set of all Fredholm operators.

2.16. Theorem. *If H is a complex Hilbert space, then* $\operatorname{Fred}_n(H)$ *is arcwise connected.*

The proof uses:

2.17. Theorem. *Let H be a complex Hilbert space. Then the group $\mathcal{L}(H)^{-1}$ of invertible elements in H is arcwise connected.*

Proof of Theorem 2.16. In view of Lemma 2.15 it is sufficient to show that any Fredholm operator A of index zero can be connected to I by a continuous path. Indeed, choose orthogonal complements X of $\ker A$ and Y of $\operatorname{im} A$. Then $A : X \to \operatorname{im} A$ is invertible, and there is an isomorphism $T : \ker A \to Y$. Clearly, $A + T$ is invertible. By Theorem 2.17 it can be connected to I by a continuous path in $\mathcal{L}(H)^{-1} \subseteq \operatorname{Fred}_0(H)$. For $0 \leq t \leq 1$ on the other hand, $t \mapsto A + tT$ defines a continuous path in $\operatorname{Fred}_0(H)$, since T has finite rank. ◁

2.18. Remark. For a Banach space X it was shown by Douady [7] that, in general, $\operatorname{Fred}_0(X)$ is not connected. A particular example is $X = c_0(\mathbb{N}) \oplus l^2(\mathbb{N})$. Note that, *a fortiori*, $\mathcal{L}(X)^{-1}$ then is not connected.

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[3]A function $f : \mathcal{T} \to \mathbb{R}$ is upper semi-continuous at t_0, if for every $\epsilon > 0$ there exists a neighborhood U of t_0 such that $f(t) \leq f(t_0) + \epsilon$ for all $t \in U$