10. The Cohomological Index Formula

10.1. Outline. We consider a class $x \in K_c(TM)$ for the compact manifold M. Our task is to compute $\operatorname{ind}_t x$ in cohomological terms. We recall that one starts with the embeddings of M into $\mathbb{R}^K$ for some large K.

$$\operatorname{ind}_t x : K_c(TM) \xrightarrow{i_!} K_c(TN) \xrightarrow{F} K_c(T\mathbb{R}^K) \xrightarrow{(j_!)^{-1}} \mathbb{Z}$$

is given by the Thom isomorphism $i_!$ associated with the embedding of M into the normal bundle N to M in $\mathbb{R}^K$, the identification of TN with an open neighborhood of TM in TN, the extension map F and the inverse of the Thom isomorphism associated with the embedding $j : \{0\} \hookrightarrow \mathbb{R}^K$.

We therefore consider the diagram

$$\begin{array}{ccccccc} K_c(TM) & \xrightarrow{i_!} & K_c(TN) & \xrightarrow{F} & K_c(\mathbb{R}^K) & \xleftarrow{j_!} & K(T0) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H_c^{even}(TM) & \xrightarrow{i_*} & H_c^{even}(TN) & \xrightarrow{F_*} & H_c^{even}(T\mathbb{R}^K) & \xleftarrow{j_*} & H_c^{even}(0). \end{array}$$

with the Chern character and the cohomology maps i_*, F_* and j_* corresponding to $i_!, F$ and $j_!$.

As a preliminary result we note:

10.2. Lemma. *Let X be a compact space and E a vector bundle over X with projection $p : E \to X$ and total space N. Denote by $\tilde{E}$ the lift of E to N.*

Recall that the Bott generator β_E is given by the class of

$$\beta_E = (\Lambda^{even}(E^*), \Lambda^{odd}(E^*), b(z))$$

in $K_c(N)$, where $b(z) = \varepsilon(z) + \varepsilon^*(z)$.

We have

$$\begin{aligned} \text{(1)} \qquad \operatorname{ch} \beta_E &= \operatorname{ch}(\Lambda^{even}(E^*)) - \operatorname{ch}(\Lambda^{odd}(E^*)) \\ &= \det \tilde{\omega} \det \frac{1 - e^{-\tilde{\omega}}}{\tilde{\omega}} = \det \tilde{\omega} \operatorname{Td}^{-1}(\tilde{E}). \end{aligned}$$

Here $\tilde{\omega}$ is the normalized curvature form associated with a special connection which we will construct in the proof.

Moreover, we obtain

$$p_! \operatorname{ch} \beta_E = \operatorname{Td}^{-1}(E).$$

Proof. Let ∂ be a hermitian connection on E and Γ the connection form in local coordinates. This defines a connection on the lifting $\tilde{E}$ of E to N with the same connection form Γ, the so-called lifting of ∂. We also denote it by ∂. The bundle $\tilde{E}$ has the section $\nu = z/|z|$, defined for $z \neq 0$. Here z is the fiber variable in E.

We next define a connection $\tilde{\partial}$ for $\tilde{E}$ by letting

$$\tilde{\partial} u = \partial u + \rho(-\partial\nu\langle u, \nu\rangle + \nu\langle u, \partial\nu\rangle + \langle \partial\nu, \nu\rangle u).$$

It satisfies $\tilde{\partial}\nu = 0$ for $|z| \geq 1$. Differentiating this identity we conclude that $\tilde{\Omega}\nu = 0$, so that also $\det \tilde{\omega} = 0$ for $|z| \geq 1$. In particular, the m-th Chern form yields a cohomology class in $H_c^{2m}(N)$.

It can be shown that $p_* \det \tilde{\omega} = 1$, so that $\det \tilde{\omega} = U_E$, the Thom generator.

Now for the computation of $p_! \operatorname{ch} \beta_E$. With the above connection $\tilde{\partial}$ associate connections on E^* and the exterior powers $\Lambda^k(E^*)$. From the fact that $\tilde{\partial}\nu = 0$ one obtains $\tilde{\partial}b(z) = 0$ (CHK). By definition, we therefore have

$$\operatorname{ch} \beta_E = \operatorname{tr}(e^{\tilde{\omega}_0}) - \operatorname{tr}(e^{\omega_1}),$$

where $\tilde{\omega}_0$ and ω_1 are the normalized curvatures associates with $\tilde{\partial}$ in $\Lambda^{even}(E^*)$ and $\Lambda^{odd}(E^*)$, respectively. From Lemma 7.22(c) we obtain (1).

We know already that $\det \tilde{\omega}$ is a form with compact support on N which defines the generator $U_E \in H^{2m}_{comp}(N)$ since $p_*(\det \tilde{\omega}) = 1$.

In the factor $\det \frac{1-e^{-\tilde{\omega}}}{\tilde{\omega}}$ we can replace $\tilde{\omega}$ by ω, the normalized curvature associated with ∂ without changing the cohomology class $H(N)$ nor the class of all the products in $H_{comp}(N)$. Since the form ω does not contain the differentials dz^j and $d\bar{z}^j$ (in fact, by the computation in 7.22, $\det(\frac{\omega}{I-\exp(-\omega)})$ is precisely the Todd class for E) and $p_*(\det \tilde{\omega}) = 1$, integration over the fibers yields

$$p_*(\operatorname{ch} \beta_E) = \operatorname{Td}^{-1}(E).$$

□

10.3. Proposition. *Let E be a complex vector bundle of rank n over the space X, $i : X \to E$ the embedding of X into E as the zero section, $i_!$ and $\tilde{i}_!$ the Thom isomorphisms in K-theory and cohomology, respectively, and $p : E \to X$ the base point projection. Consider the diagram*

$$\begin{array}{ccc} K_c(X) & \xrightarrow{\;i_!\;} & K_c(E) \\ \Big\downarrow{\scriptstyle \operatorname{ch}} & & \Big\downarrow{\scriptstyle \operatorname{ch}} \\ H^{even}_c(X) & \xrightarrow{\;\tilde{i}_!\;} & H^{even}_c(E). \end{array}$$

Then, for $x \in K_c(X)$,

$$(\tilde{i}_!)^{-1} \operatorname{ch} i_!(x) = \operatorname{Td}(E)^{-1} \wedge \operatorname{ch} x.$$

Proof. Thom's isomorphism theorem 8.10 states that $i_! x = \beta_E\, p^! x$ is given by multiplication of the class of the pull-back with the Bott generator β_E. Then

$$\begin{aligned} (\tilde{i}_!)^{-1} \operatorname{ch} i_!(x) &= (\tilde{i}_!)^{-1} \operatorname{ch}(\beta_E\, p^! x) \\ &\overset{8.8}{=} (\tilde{i}_!)^{-1} (\operatorname{ch} \beta_E \wedge \operatorname{ch} p^! x) \qquad (\operatorname{ch} \text{ is module homomorphism}) \\ &= ((\tilde{i}_!)^{-1} \operatorname{ch} \beta_E) \wedge \operatorname{ch} x) \qquad (\tilde{i}_! \operatorname{ch} = \operatorname{ch} p^!) \\ &\overset{10.2}{=} \operatorname{Td}(E)^{-1} \wedge \operatorname{ch} x. \end{aligned}$$

□

10.4. Corollary. *Consider the scrunch map*

$$q : T\mathbb{R}^K \cong \mathbb{R}^K \oplus \mathbb{R}^K \cong \mathbb{C}^K \to \{0\}.$$

We consider $\mathbb{C}^K$ as a complex vector bundle over $\{0\}$ and let $j \hookrightarrow \mathbb{C}^K$ be the inclusion map. Apply Proposition 10.3 for a class $e \in K(0)$. Then $(\tilde{j}_!)^{-1} \operatorname{ch}(j_! e) = \operatorname{Td}(\mathbb{C}^K)^{-1} \operatorname{ch} e$. Now $\mathbb{C}^K$ is a trivial bundle. The curvature *is zero and therefore* $\mathrm{Td}(\mathbb{C}^K) = 1 = \mathrm{Td}(\mathbb{C}^K)^{-1}$*. Moreover,* $(\tilde{j})^{-1} = q_!$ *is simply integration, and the Chern character* $ch : K(0) \to H^0(0)$ *is the map* $\mathbb{Z} \ni z \mapsto z$*. For* $u = j_! e \in K_c(\mathbb{C}^K)$ *we therefore find*

$$\int_{T\mathbb{R}^K} \mathrm{ch}\, u = e \quad \textit{considered as an integer.}$$

10.5. Corollary. *Let* V *be a real vector bundle over* M*,* $p : V \to M$ *the projection. Consider the induced bundle* TV *over* TM*. Here, we have a natural isomorphism* $TV \cong p^*V \oplus p^*V$ *(first term in the base, second in the fiber). Therefore* TV *naturally has a complex structure as a vector bundle over* TM*, and we can identify it with the complex bundle* $V \otimes \mathbb{C}$*. We therefore can apply Proposition 10.3 with* TM *in the role of* X *and* $TV \cong V \otimes \mathbb{C}$ *in the role of* E *and obtain*

$$(\tilde{i}_!)^{-1}\,\mathrm{ch}(i_! x) = \mathrm{Td}(V \otimes \mathbb{C})^{-1}\,\mathrm{ch}\, x, \quad x \in K_c(TM).$$

Now we recall from the proof of the Todd isomorphism Theorem 8.10 that $(\tilde{i}_!)^{-1} = p_!$ *and that* $p_!$ *is integration over the fiber. Integrating over all of* TV *we obtain*

$$\int_{TV} \mathrm{ch}(i_! x) = \int_{TM} p_!\,\mathrm{ch}(i_! x) = \int_{TM} \mathrm{Td}^{-1}(V \otimes \mathbb{C})\,\mathrm{ch}\, x.$$

10.6. Proof of the cohomological formula. We recall the construction of the map $f_!$ associated with the embedding $f : X \hookrightarrow \mathbb{R}^K$: It is $F \circ i_!$, i.e. the Thom isomorphism $i_! : K_c(TM) \to K_c(TN)$ for the normal bundle N to the embedding of X into $\mathbb{R}^K$ followed by the inclusion F, where now TN is identified with an open neighborhood of X in $T\mathbb{R}^K$. The fact that the Chern character behaves naturally under the embedding $TN \hookrightarrow T\mathbb{R}^K$ shows that

$$(1)\quad \int_{T\mathbb{R}^K} \mathrm{ch}(f_! x) = \int_{T\mathbb{R}^K} \mathrm{ch}(F \circ i_! x) = \int_{TN} \mathrm{ch}(i_! x), \quad x \in K_c(TM).$$

According to Corollary 10.5, with N in the role of V,

$$(2)\quad \int_{TN} \mathrm{ch}(i_! x) = \int_{TM} \mathrm{Td}^{-1}(N \otimes \mathbb{C})\,\mathrm{ch}\, x.$$

We next note that $TM \oplus N$ is a trivial bundle over X. The Todd class of a trivial bundle is 1, since the curvature is zero. Since the Todd class is multiplicative, we conclude that

$$\mathrm{Td}^{-1}(N \otimes \mathbb{C}) = \mathrm{Td}(TM \otimes \mathbb{C}) =: \mathrm{Td}(M),$$

so that the right hand side of (2) equals

$$(3)\quad \int_{TM} \mathrm{Td}(M)\,\mathrm{ch}\, x.$$

Now we apply this to the class $x = [\sigma(P)] = [(\pi^* E^0, \pi^* E^1, \sigma)]$ in $K_c(TM)$ associated with a pseudodifferential operator

$$P : C^\infty(M, E^0) \to C^\infty(M, E^1).$$

Its analytic index is $\mathrm{ind}\, P$. The topological index is (see the diagram in 10.1) $\mathrm{ind}_t \circ (j_!)^{-1} \circ f_!$.

We can then apply Corollary 10.4 for the case, where $j_! e = f_! x$; and combine it with the above computation and the various definitions to see that

$$\begin{aligned} \operatorname{ind} P = e = \int_{T\mathbb{R}^K} \operatorname{ch} j_! e \\ \overset{j_!e=f_!x}{=} \int_{T\mathbb{R}^K} \operatorname{ch}(f_! x) \overset{(1),(2),(3)}{=} \int_{TM} \operatorname{Td}(M) \operatorname{ch} x \\ = \int_{TM} \operatorname{Td}(M) \operatorname{ch}[\sigma(P)]. \end{aligned}$$

10.7. Corollary. *On an odd-dimensional compact manifold, the index of every elliptic differential operator is zero.*

Proof. Consider the diffeomorphism $c : TM \to TM$ given by $(x, v) \mapsto (x, -v)$. If the dimension of M is odd, then c changes the orientation.

If P is an elliptic differential operator of order m, then the principal symbol $\sigma(P)$ is an m-homogeneous function, and therefore $\sigma(P)(x, -\xi) = (-1)^m P(x, \xi)$. Hence $c^*\sigma(P) = (-1)^m \sigma(P)$. Since $\sigma(P)$ and $\sigma(P)(\cdot, -\cdot)$ are homotopic, e.g. by $\sigma(t, P) = e^{i\pi t}\sigma(P)$. Reducing the order to zero results in multiplying the function $\sigma(P)(x, \xi)$ by a strictly positive, $-m$-homogeneous factor, say $\lambda(x, \xi)$. It is then clear that the triples $(\pi^* E^0, \pi^* E^1, \sigma(P)/\lambda)$ and $(\pi^* E^0, \pi^* E^1, -\sigma(P)/\lambda)$ define the same class in K-theory, and therefore (note that the Todd class only depends on the x-variables and therefore does not feel the action of c)

$$\begin{aligned} \operatorname{ind} P &= \int_{TX} \operatorname{ch}([\sigma(P)]) \operatorname{Td}(X) \\ &= \int_{c^*TX} c^*(\operatorname{ch}([\sigma(P)]) \operatorname{Td}(X)) \\ &= -\int_{TX} (\operatorname{ch}([\sigma(P)]) \operatorname{Td}(X)) \quad \text{change in orientation; invariance of K-class} \\ &= -\operatorname{ind} P \end{aligned}$$

□

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