1. Prerequisites

In the sequel, X, Y, Z and W will denote Banach spaces. We denote by $\mathcal{L}(X,Y)$ the continuous linear operators from X to Y. It is well-known that an operator A is continuous, if and only if it is bounded, i.e., there exists a constant c such that

$$\|Ae\| \le c\|e\| \text{ for all} e \in X.$$

We write $\mathcal{K}(X,Y)$ for the subspace of compact operators (more below) and $\mathcal{F}(X,Y)$ for the space of operators of finite rank.

We will need as few theorems from functional analysis:

1.1. Neumann series. Let $A \in \mathcal{L}(X)$ with $\|A\| < 1$. Then $I + A$ is invertible; in fact $(I + A)^{-1} = \sum_0^\infty A^j$.

1.2. Open mapping theorem. Let $A \in \mathcal{L}(X,Y)$ be surjective. Then A is an open mapping, i.e. the image of every open set is open.

1.3. Automatic continuity of inverses. Let $A \in \mathcal{L}(X,Y)$ be invertible. Then the inverse is automatically continuous, i.e. $A^{-1} \in \mathcal{L}(Y,X)$. This is a consequence of the fact that, given an open set in X, the preimage under A^{-1} is its image under A, which is open by Theorem 1.2.

1.4. Comparable norms on Banach spaces are equivalent. Suppose a space X carries two norms, $\|\cdot\|$ and $\|\cdot\|'$. Moreover, assume it is a Banach space for both norms and there exists a constant c such that

$$\|e\|' \le c\|e\| \text{ for all } e \in X. \tag{1}$$

Then there also exists a constant $c' > 0$ such that $c'\|e\| \le \|e\|' \le c\|e\|$.

In fact, inequality (1) shows that the identity is a continuous map from $(X, \|\cdot\|)$ to $(X, \|\cdot\|')$. By Theorem 1.3 the inverse (here again the identity) is also continuous, which gives the second inequality.

1.5. Compact operators. We call an operator $K \in \mathcal{L}(X,Y)$ compact, if it maps bounded sets into compact sets. By linearity it suffices that the closure $\overline{K(B(0,1))}$ of the image of the unit ball is compact.

Clearly, every operator of finite rank is compact. For $K \in \mathcal{K}(X,Y)$, $A \in \mathcal{L}(Y,Z)$, and $B \in \mathcal{L}(W,X)$ the operators AK and KB are compact. Moreover, the space $\mathcal{K}(X,Y)$ of is a closed in $\mathcal{L}(X,Y)$. In particular, $\mathcal{K}(X)$ is a closed ideal in $\mathcal{L}(X)$. The quotient $\mathcal{L}(X)/\mathcal{K}(X)$ is called the Calkin algebra.

1.6. Compactness of the unit ball. The closed unit ball $\overline{B(0,1)}$ in a Banach space X is compact if and only if the space is finite dimensional. By linearity the statement holds for every closed ball.

Proof. It is known that the closed unit ball in a finite dimensional Banach space is compact.

Conversely suppose that it is compact. Then it is precompact, i.e. we find $x_1, \ldots, x_m \in B(0,1)$ such that

$$B(0,1) \subseteq \left(x_1 + B(0,\frac{1}{2})\right) \cup \ldots \cup \left(x_m + B(0,\frac{1}{2})\right).$$

Let $Y = \text{span}\,\{x_1, \dots, x_m\}$. Then

$$(1) \qquad B(0,1) \subseteq Y + B(0, \frac{1}{2}).$$

So $B(0, \frac{1}{2}) \subseteq Y + B(0, \frac{1}{4})$. Inserting this into (1) shows: $B(0,1) \subseteq Y + Y + B(0, \frac{1}{4}) = Y + B(0, \frac{1}{4})$. By iteration: $B(0,1) \subseteq Y + B(0, \varepsilon)$ for every $\varepsilon > 0$, hence $B(0,1) \subseteq \overline{Y} = Y$, and $X = Y$ is at most m-dimensional. □

1.7. Quotient norm. Let X' be a closed subspace of X. Then we can form the quotient space X/X' and endow it with the norm $\|\,[e]\,\| = \inf\{\|e + e'\| : e' \in X'\}$.

1.8. Definition. Let X_1, X_2 be subspaces of a Banach space X with $X = X_1 \oplus X_2$. We say that the sum is *topologically direct* if the canonical algebraic isomorphism $(x_1, x_2) \mapsto x_1 + x_2$ from $X_1 \times X_2$ to X is a homeomorphism. Equivalently we may ask that the associated projection $P_{X_1} : X \to X_1$ (or $P_{X_2} : X \to X_2$) is continuous. Note that $(x_1, x_2) \mapsto x_1 + x_2$ is always continuous, since $\|x_1 + x_2\| \le \|x_1\| + \|x_2\|$.

1.9. Lemma. *Let X_1 and X_2 be closed subspaces of the Banach space X with $X = X_1 \oplus X_2$. Then the sum is topologically direct.*

Proof. Being closed subspaces of a Banach space, X_1 and X_2 are complete with the induced norm. Since the decomposition $x = x_1 + x_2$ of an element $x \in X$ into $x_1 \in X_1$ and $x_2 \in X_2$ is unique, we may define a norm $\|\cdot\|'$ on X by $\|x\|' = \|x_1\| + \|x_2\|$. Since $\|x\| = \|x_1 + x_2\| \le \|x_1\| + \|x_2\| = \|x\|'$, Theorem 1.4 shows the equivalence of both. So $\|x_1\| + \|x_2\| \le C\|x\|$, and the sum is topologically direct. □

1.10. Lemma. *Let X_1 be a finite-dimensional and X_2 a closed subspace of X, and suppose that $X_1 \cap X_2 = \{0\}$. Then $X_1 \oplus X_2$ is a closed subspace of X, and the sum is topologically direct.*

Proof. Let $\pi : X \to X/X_2$ be the canonical epimorphism. Then $\pi(X_1)$ is a finite-dimensional subspace of X/X_2, hence closed. The continuity of π implies that $X_1 \oplus X_2 = \pi^{-1}(\pi(X_1))$ is also closed, thus a Banach space. Lemma 1.9 concludes the proof. □

1.11. Lemma. *Every finite-dimensional subspace Y of a normed space X is closed; it even is the image of a continuous projection.*

Proof. Given a basis $\{x_1, \dots, x_n\}$ of Y choose $x_1', \dots, x_n' \in Y'$ with $x_j'(x_k) = \delta_{jk}$. By Hahn and Banach's theorem we may extend the x_j' to continuous functionals on X. Now let

$$Px = \sum_{j=1}^{n} x_j'(x) x_k.$$

P is continuous, since all x_j' are. Moreover, $PX = Y$, and it is easily verified that $P^2 = P$. Finally, $Y = \operatorname{im} P = \ker(I - P)$ is closed. □

1.12. Arzelà-Ascoli Theorem. Let $\mathcal{T}$ be a compact Hausdorff space. A subset $\mathscr{F}$ of $\mathscr{C}(\mathcal{T})$ is relative compact (i.e. has compact closure), if and only if it is pointwise bounded and equicontinuous.

Recall that a set $\mathscr{F}$ of complex-valued functions on $\mathcal{T}$ is equicontinuous provided that for every x_0 in $\mathcal{T}$ and every $\epsilon > 0$ there exists a neighborhood U of x_0 such that

$$|f(x) - f(x_0)| < \epsilon \text{ for all } x \in U \text{ and all } f \in \mathscr{F}.$$