Reflection positivity, duality, and spectral theory

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History

Since early work in mathematical physics, starting in the 1970ties, and initiated by A. Jaffe, and by K. Osterwalder and R. Schrader, the subject of reflection positivity has had an increasing influence on both non-commutative harmonic analysis, and on duality theories for spectrum and geometry. In its original form, the Osterwalder-Schrader idea served to link Euclidean field theory to relativistic quantum field theory. It has been remarkably successful; especially in view of the abelian property of the Euclidean setting, contrasted with the non-commutativity of quantum fields. Osterwalder-Schrader and reflection positivity have also become a powerful tool in the theory of unitary representations of Lie groups. Co-authors in this subject include G. Olafsson, and K.-H. Neeb.

Outline

We consider reflection-positivity (Osterwalder-Schrader positivity, O.S.-p.) as it is used in the study of renormalization questions in physics. In concrete cases, this refers to specific Hilbert spaces that arise before and after the reflection.

► Geometric properties connected with OS axioms

► OS.p. $\iff$ a triple projections $(E_0, E_\pm)$ in a given Hilbert space $\mathscr{H}$

► Markov property

► Probabilistic counterpart: OS-positive processes

Setting

- $\mathscr{H}$: a given Hilbert space;
- $U, \theta : \mathscr{H} \to \mathscr{H}$, unitary operators, s.t.

$$\theta^2 = I_{\mathscr{H}}, \ \theta^* = \theta, \text{ and} \tag{0.1}$$
$$\theta U \theta = U^*. \tag{0.2}$$

- There exists a closed subspace $\mathscr{H}_+ \subset \mathscr{H}$ s.t.

$$U \mathscr{H}_+ \subset \mathscr{H}_+, \text{ and} \tag{0.3}$$

$$\langle h_+, \theta h_+ \rangle \geq 0, \ \forall h_+ \in \mathscr{H}_+. \tag{0.4}$$

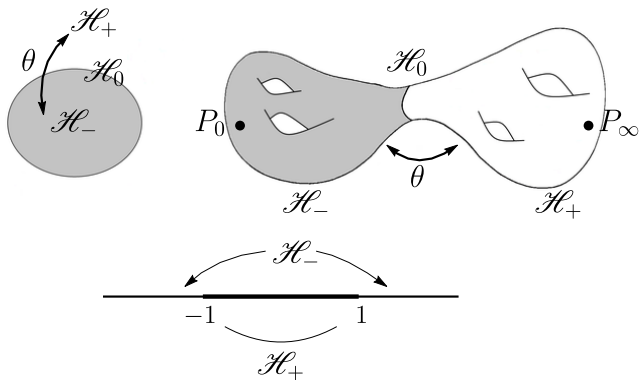**Note.** $\theta = 2P - I_{\mathscr{H}}$, where P = projection onto $\{h \in \mathscr{H} \mid \theta h = h\}$.

Figure 0.1: (a) The complex plane, inside and outside of the disk. (b) The Schottky double S of a bordered Riemann surface T with boundary ∂T. (c) The real line, inside and outside of a fixed interval.

1. Unitary Representations of Lie Groups with Reflection Symmetry

We consider a class of unitary representations of a Lie group G which possess a certain reflection symmetry defined as follows.

Notation. If π is a representation of G in some Hilbert space H, we introduce the following three structures:

1. $\tau \in Aut(G)$ of period 2;

2. $J : \mathrm{H} \to \mathrm{H}$ is a unitary operator of period 2 such that $J\pi(g)J^* = \pi(\tau(g))$, $g \in G$ (this will hold if π is of the form $\pi_+ \oplus \pi_-$ with π_+ and $\pi_- \circ \tau$ unitarily equivalent); it will further be assumed that there is a closed subspace $\mathrm{K}_0 \subset \mathrm{H}$ which is invariant under $\pi(H)$, $H = G^\tau$, or more generally, an open subgroup of G^τ;

3. positivity is assumed in the sense that $\langle v, J(v) \rangle \geq 0$, $v \in \mathrm{K}_0$.

Definition 1.1. A unitary representation π acting on a Hilbert space $H(\pi)$ is said to be *reflection symmetric* if there is a unitary operator $J : H(\pi) \to H(\pi)$ such that

  1. $J^2 = id$.

  2. $J\pi(g) = \pi(\tau(g))J$, $\quad g \in G$.

Note. If (1) holds then π and $\pi \circ \tau$ are equivalent. Furthermore, generally from (2) we have $J^2\pi(g) = \pi(g)J^2$. Thus, if π is irreducible, then we can always renormalize J such that (1) holds. Let $H = G^\tau = \{g \in G \mid \tau(g) = g\}$ and let $\mathfrak{h}$ be the Lie algebra of H. Then $\mathfrak{h} = \{X \in \mathfrak{g} \mid \tau(X) = X\}$. Define $\mathfrak{q} = \{Y \in \mathfrak{g} \mid \tau(Y) = -Y\}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$.

Reflection Symmetry for Complementary Series

Theorem 1.2 (Jor-Ólafsson). Assume that G/H is non-compactly causal and such that there exists a $w \in K$ such that $Ad(w)|_{\mathfrak{a}} = -1$. Let π_ν be a complementary series such that $\nu \leq L_{\text{pos}}$. Let C be the minimal H-invariant cone in $\mathfrak{q}$ such that $S(C)$ is contained in the contraction semigroup of $HP_{\max}$ in $G/P_{\max}$. Let Ω be the bounded realization of $H/H \cap K$ in $\overline{\mathfrak{n}}$. Let $J(f)(x) := f(\tau(x)w^{-1})$. Let K_0 be the closure of $\mathcal{C}_c^\infty(\Omega)$ in H_ν. Then the following holds:

Theorem 1.2, cont.

1. $(G, \tau, \pi_\nu, C, J, K_0)$ satisfies the positivity conditions (PR1)–(PR2).

2. π_ν defines a contractive representation $\tilde{\pi}_\nu$ of $S(C)$ on K such that $\tilde{\pi}_\nu(\gamma)^* = \tilde{\pi}_\nu(\tau(\gamma)^{-1})$.

3. There exists a unitary representation $\tilde{\pi}_\nu^c$ of G^c such that
    (i) $d\tilde{\pi}_\nu^c(X) = d\tilde{\pi}_\nu(X) \quad \forall X \in \mathfrak{h}$.
    (ii) $d\tilde{\pi}_\lambda^c(iY) = i\, d\tilde{\pi}_\lambda(Y) \quad \forall Y \in C$.

2. Osterwalder-Schrader Axioms-Wightman Axioms

Definition 2.1. A closed convex cone $C \subset \mathfrak{q}$ is *hyperbolic* if $C^o \neq \emptyset$ and if $\operatorname{ad} X$ is semisimple with real eigenvalues for every $X \in C^o$.

We will assume the following for (G, π, τ, J):

- **(PR1)** π is reflection symmetric with reflection J;
- **(PR2)** there is an H-invariant hyperbolic cone $C \subset \mathfrak{q}$ such that $S(C) = H \exp C$ is a closed semigroup and $S(C)^o = H \exp C^o$ is diffeomorphic to $H \times C^o$;
- **(PR3)** there is a subspace $0 \neq \mathbf{K}_0 \subset \mathbf{H}(\pi)$ invariant under $S(C)$ satisfying the positivity condition

$$\langle v, v \rangle_J := \langle v, J(v) \rangle \geq 0, \quad \forall v \in \mathbf{K}_0.$$

Theorem 2.2 (Jor-Ólafsson). Assume that $(\pi, C, \mathbf{H}, J)$ satisfies (PR1)–(PR3). Then the following hold:

1. $S(C)$ acts via $s \mapsto \tilde{\pi}(s)$ by contractions on $\mathbf{K}$.

2. Let G^c be the simply connected Lie group with Lie algebra $\mathfrak{g}^c$. Then there exists a unitary representation $\tilde{\pi}^c$ of G^c such that $d\tilde{\pi}^c(X) = d\tilde{\pi}(X)$ for $X \in \mathfrak{h}$ and $i\, d\tilde{\pi}^c(Y) = d\tilde{\pi}(iY)$ for $Y \in C$.

3. The representation $\tilde{\pi}^c$ is irreducible if and only if $\tilde{\pi}$ is irreducible.

Definition 2.3. Let W be a G-invariant cone in $\mathfrak{g}$. We denote the set of all unitary representations π of G with $W \subset W(\pi)$ by $\mathcal{A}(W)$. A unitary representation π is called W-*admissible* if $\pi \in \mathcal{A}(W)$.

It turns out that the irreducible representations in $\mathcal{A}(W)$ are *highest weight representations.* A $(\mathfrak{g}^c, K^c)$-module is a complex vector space $\mathbf{V}$ such that

1) $\mathbf{V}$ is a $\mathfrak{g}^c$-module.

2) $\mathbf{V}$ carries a representation of K^c, and the span of $K^c \cdot v$ is finite-dimensional for every $v \in \mathbf{V}$.

3) For $v \in \mathbf{V}$ and $X \in \mathfrak{k}^c$ we have

$$X \cdot v = \lim_{t \to 0} \frac{\exp(tX) \cdot v - v}{t}.$$

4) For $Y \in \mathfrak{g}^c$ and $k \in K^c$ the following holds for every $v \in \mathbf{V}$:

$$k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot [k \cdot v].$$

Definition 2.4. Let $\mathbf{V}$ be a $(\mathfrak{g}^c, K^c)$-module. Then $\mathbf{V}$ is a highest-weight module if there exists a nonzero element $v \in \mathbf{V}$ and a $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ such that

1) $X \cdot v = \lambda(X)v$ for all $X \in \mathfrak{t}$.

2) There exists a positive system Δ^+ in Δ such that $\mathfrak{g}_{\mathbb{C}}^c(\Delta^+) \cdot v = 0$.

3) $\mathbf{V} = U(\mathfrak{g}^c) \cdot v$.

The element v is called a primitive element of weight λ.

Theorem 2.5 (Jor-Ólafsson). Let $\rho \in \mathcal{A}(W)$ be irreducible. Then the corresponding $(\mathfrak{g}^c, K^c)$-module is a highest-weight module and equals $U(\mathfrak{p}^-)\mathbf{W}^\lambda$. In particular, every weight of $\mathbf{V}_{K^c}$ is of the form

$$\nu - \sum_{\alpha \in \Delta(\mathfrak{p}^+, \mathfrak{t}_\mathbb{C})} n_\alpha \alpha \,.$$

Furthermore, $\langle \nu, H_\alpha \rangle \leq 0$ for all $\alpha \in \Delta_p^+$.

3. Reflection Positive Stochastic Processes Indexed by Lie Groups

Definition 3.1. A *reflection positive Hilbert space* is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, where $\mathcal{E}$ is a Hilbert space, θ a unitary involution and $\mathcal{E}_+$ a closed subspace which is θ-*positive* in the sense that the hermitian form $\langle v, w \rangle_\theta := \langle \theta v, w \rangle$ is positive semidefinite on $\mathcal{E}_+$. For a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$, let

$$\mathcal{N} := \{u \in \mathcal{E}_+ \colon \langle \theta u, u \rangle = 0\}$$

and let $\widehat{\mathcal{E}}$ be the completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\theta$. Let $q \colon \mathcal{E}_+ \to \hat{\mathcal{E}}, v \mapsto q(v) = \hat{v}$ be the canonical map. Then

$$\mathcal{E}_+^\theta := \{v \in \mathcal{E}_+ \colon \theta v = v\}$$

is the maximal subspace of $\mathcal{E}_+$ on which q is isometric.

Definition 3.2. Let $(\mathcal{E}, \mathcal{E}_+, \theta)$ be a reflection positive Hilbert space. If $\mathcal{E}_0 \subseteq \mathcal{E}_+^\theta$ is a closed subspace, $\mathcal{E}_- := \theta(\mathcal{E}_+)$, and $E_0, E_\pm$ the orthogonal projections onto $\mathcal{E}_0$ and $\mathcal{E}_\pm$, then we say that $(\mathcal{E}, \mathcal{E}_0, \mathcal{E}_+, \theta)$ is *of Markov type* if

$$E_+ E_0 E_- = E_+ E_-. \tag{3.1}$$

Proposition 3.3. Let $(U_g)_{g \in G}$ be a reflection positive unitary representation of (G, S, τ) on $(\mathcal{E}, \mathcal{E}_+, \theta)$, let $\mathcal{E}_0 \subseteq (\mathcal{E}_+)^{\theta}$ be a subspace and $\Gamma = q|_{\mathcal{E}_0} \colon \mathcal{E}_0 \to \widehat{\mathcal{E}}$. If $(\mathcal{E}, \mathcal{E}_0, \mathcal{E}_+, \theta)$ is of Markov type, then the following assertions hold:

(i) The reflection positive function $\varphi \colon G \to B(\mathcal{E}_0)$, $\varphi(g) := E_0 U_g E_0$, is multiplicative on S.

(ii) $\varphi(s) = \Gamma^* \hat{U}_s \Gamma$ for $s \in S$, i.e., Γ intertwines $\varphi|_S$ with $\hat{U}$.

Reconstruction Theorem

Theorem 3.4 (Jorgensen, Neeb, Ólafsson). Let (G, τ) be a symmetric Lie group and $S \subseteq G$ be a $\sharp$-invariant subsemigroup satisfying $G = S \cup S^{-1}$. Then every positive semigroup structure for (G, S, τ) is associated to some (G, S, τ)-probability space $((Q, \Sigma, \mu), \Sigma_0, U, \theta)$.

Standard path space structures for locally compact groups

Theorem 3.5 (Jorgensen, Neeb, Ólafsson). Suppose that Q is a second countable locally compact group. Let μ be the measure on $Q^{\mathbb{R}}$ corresponding to the symmetric convolution semigroup $(\mu_t)_{t \geq 0}$ of probability measures on Q and the measure ν on Q for which the operators $P_t f = f * \mu_t$ define a positive semigroup structure on $L^2(Q, \nu)$. Then the translation action $(U_t \omega)(s) := \omega(s - t)$ on $P(Q) = Q^{\mathbb{R}}$ is measure preserving and μ is invariant under $(\theta\omega)(t) := \omega(-t)$.

Theorem 3.5, cont.

We thus obtain a reflection positive one-parameter group of Markov type on $\mathcal{E} := L^2(P(Q), \mathfrak{B}^{\mathbb{R}}, \mu)$ with respect to

- $\mathcal{E}_+ := L^2(P(Q), \mathfrak{B}^{\mathbb{R}_+}, \mu)$, for which
- $\mathcal{E}_0 := \mathrm{ev}_0^*(L^2(Q, \nu)) \cong L^2(Q, \nu)$ and
- $\hat{\mathcal{E}} \cong L^2(Q, \nu)$ with $q(F) = E_0 F$ for $F \in \mathcal{E}_+$.

We further have

$$E_0 U_t E_0 = P_t \qquad \text{holds for} \quad P_t f = f * \mu_t,$$

so that the U-cyclic subrepresentation generated by $\mathcal{E}_0$ is a unitary dilation of the hermitian one-parameter semigroup $(P_t)_{t \geq 0}$ on $L^2(Q, \nu)$.

4. Reflection Positivity and Spectral Theory

Our focus is a comparative study of the associated spectral theory, now referring to the canonical operators in these two Hilbert spaces. Indeed, the inner product which produces the respective Hilbert spaces of quantum states changes, and comparisons are subtle.
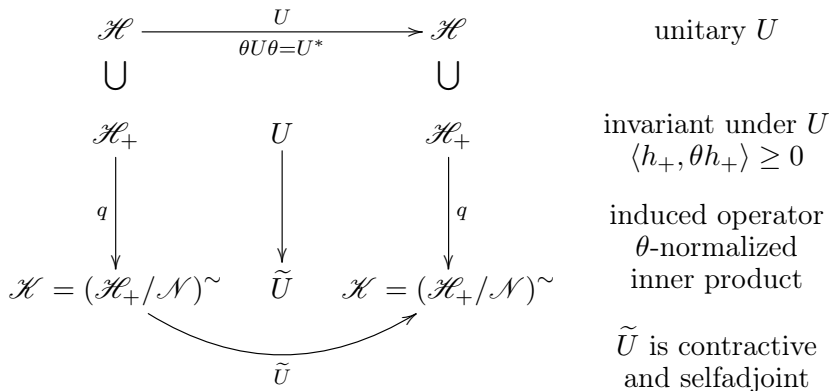
Figure 4.1: Reflection positivity. A unitary operator U transforms into a selfadjoint contraction $\widetilde{U}$.

An example ([JO98, JO00])

Let $0 < s < 1$ be given, and let $\mathscr{H} = \mathscr{H}_s$ be the Hilbert space whose norm $\|f\|_s$ is given by

$$\|f\|_s^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(x)} \, |x - y|^{s-1} \, f(y) \, dxdy. \qquad (4.1)$$

Let $a \in \mathbb{R}_+$ be given, and set

$$(U(a) f)(x) = a^{s+1} f(a^2 x). \qquad (4.2)$$

Let $\mathscr{H}_+$ be the closure of $C_c(-1, 1)$ in $\mathscr{H}_s$ relative to the norm $\|\cdot\|_s$. It is then immediate that $U(a)$, for $a > 1$, leaves $\mathscr{H}_+$ invariant, i.e., it restricts to a semigroup of isometries $\{U(a)\, ;\, a > 1\}$ acting on $\mathscr{K}_s$.

Setting

$$(\theta f)(x) = |x|^{-s-1} f\left(\frac{1}{x}\right), \quad x \in \mathbb{R} \setminus \{0\}, \qquad (4.3)$$

we check that θ is then a period-2 unitary in $\mathscr{H}_s$, and that

$$\theta U(a)\theta = U(a)^* = U(a^{-1}) \qquad (4.4)$$

and

$$\langle f, \theta f \rangle_{\mathscr{H}_s} \geq 0, \quad \forall f \in \mathscr{H}_+, \qquad (4.5)$$

where $\langle \cdot, \cdot \rangle_{\mathscr{H}_s}$ is the inner product

$$\langle f_1, f_2 \rangle_{\mathscr{H}_s} := \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f_1(x)} |x - y|^{s-1} f_2(y) \, dx dy. \qquad (4.6)$$

In fact, if $f \in C_c(-1, 1)$, the expression in (4.5) works out as the following reproducing kernel integral:

$$\int_{-1}^{1} \int_{-1}^{1} \overline{f(x)} (1 - xy)^{s-1} f(y) \, dxdy. \qquad (4.7)$$

Hence up to a constant, the norm $\| \cdot \|_s$ of (4.6) may be rewritten as

$$\int_{\mathbb{R}} |\xi|^{-s} \left| \hat{f}(\xi) \right|^2 d\xi, \qquad (4.8)$$

and the inner product $\langle \, \cdot \, , \, \cdot \, \rangle_s$ as

$$\int_{\mathbb{R}} |\xi|^{-s} \overline{\hat{f}_1(\xi)} \hat{f}_2(\xi) \, d\xi. \qquad (4.9)$$

Intuitively, $\mathscr{H}_s$ consists of functions on $\mathbb{R}$ which arise as $\left(\frac{d}{dx}\right)^s f_s$ for some f_s in $L^2(\mathbb{R})$. This also introduces a degree of "non-locality" into the theory, and the functions in $\mathscr{H}_s$ cannot be viewed as locally integrable, although $\mathscr{H}_s$ for each s, $0 < s < 1$, contains $C_c(\mathbb{R})$ as a dense subspace. In fact, formula (4.8), for the norm in $\mathscr{H}_s$, makes precise in which sense elements of $\mathscr{H}_s$ are "fractional" derivatives of locally integrable functions on $\mathbb{R}$, and that there are elements of $\mathscr{H}_s$ (and of $\mathscr{K}_s$) which are not locally integrable.

Conclusion: When $\mathscr{H}_+$ and $\mathscr{K}$ are as in (4.7), then the natural contractive operator q, defined as $q(h_+) = \text{class}(h_+) = h_+ + \mathscr{N}$, is automatically 1-1, i.e., its kernel is 0.

$$\mathscr{H}_+ \xrightarrow{} \mathscr{H}_+/\mathscr{N} \xrightarrow{} (\mathscr{H}_+/\mathscr{N})^{\sim} = \mathscr{K} \qquad (4.10)$$

with q the curved arrow from $\mathscr{H}_+$.

Maximal Reflections

Definition 4.1. Let $\mathscr{H}$ be a Hilbert space and θ a reflection on $\mathscr{H}$. Let $P = proj\{x \in \mathscr{H} \, ; \, \theta x = x\}$, so that $\theta = 2P - I_{\mathscr{H}}$. Set

$$Sub_{OS}(\theta) = \{E_+ \, ; \, E_+ \text{ is a projection in } \mathscr{H} \text{ s.t. } E_+\theta E_+ \geq 0\}. \tag{4.11}$$

Note. As usual properties for projections have equivalent formulation for closed subspaces: In this case, we may identify elements in $Sub_{OS}(\theta)$ with closed subspaces $\mathscr{H}_+$ such that

$$\langle h_+, \theta h_+ \rangle \geq 0, \text{ for } \forall h_+ \in \mathscr{H}_+. \tag{4.12}$$

Set $\mathscr{H}_+ := E_+\mathscr{H}$.

Theorem 4.2 (Jor-Tian). Let $\mathscr{H}$, θ, and P be as stated, and consider the corresponding $Sub_{OS}(\theta)$ as in (4.11), or equivalently (4.12).

Then $Sub_{OS}(\theta)$ is an ordered lattice of projections, and it has the following family of maximal elements: Let $C : P\mathscr{H} \longrightarrow P^{\perp}\mathscr{H}$ be a contractive operator, and set

$$\mathscr{H}_{+}(P, C) := \{x + Cx \,;\, x \in P\mathscr{H}\}. \qquad (4.13)$$

Then $\mathscr{H}_{+}(P, C)$ is maximal in $Sub_{OS}(\theta)$, and every maximal element in $Sub_{OS}(\theta)$ has this form for some contraction $C : P\mathscr{H} \to P^{\perp}\mathscr{H}$.

Theorem 4.3 (Jor-Tian). Let $\mathscr{H}$, $\mathscr{H}_{\pm}$, θ, and U be as above, i.e., we are assuming O.S.-positivity; and further that U satisfies

$$\theta U \theta = U^*; \text{ and} \tag{4.14}$$

$$U \mathscr{H}_+ \subseteq \mathscr{H}_+ \text{ (equivalently, } E_+ U E_+ = U E_+.) \tag{4.15}$$

Let P be the projection onto $\{h \in \mathscr{H} \,;\, \theta h = h\}$, i.e., we have $\theta = 2P - I_{\mathscr{H}}$.

Theorem 4.3, cont.

1. Then
$$PUE_+ = PU^*\theta E_+. \qquad (4.16)$$

2. If $C : P\mathscr{H} \longrightarrow P^\perp\mathscr{H}$ denotes the corresponding contraction, then there is a unique operator $U_P : P\mathscr{H} \longrightarrow P\mathscr{H}$ such that $U_P = PUP$; and, if $h_+ = x + Cx$, $x \in P\mathscr{H}$, then

$$\left\| \widetilde{U}q\left(h_+\right) \right\|_{\mathscr{K}}^2 = \|U_P x\|_{\mathscr{H}}^2 - \|CU_P x\|_{\mathscr{H}}^2. \qquad (4.17)$$

3. In particular, we have

$$\|U_P x\|_{\mathscr{H}}^2 - \|CU_P x\|_{\mathscr{H}}^2 \le \|x\|_{\mathscr{H}}^2 - \|Cx\|_{\mathscr{H}}^2, \ \forall x \in P\mathscr{H}.$$

Markov vs O.-S. positivity

Definition 4.4. If E_0, $E_\pm$ are projections in $\mathscr{H}$, let $\varepsilon = (E_0, E_\pm)$, and set

$$\mathscr{E}\,(Markov) := \{(E_0, E_\pm)\,;\, E_+ E_0 E_- = E_+ E_-\}, \qquad (4.18)$$

$$\mathscr{R}\,(\varepsilon) := \{\theta \in Ref\,(\mathscr{H})\,;\, \theta E_0 = E_0, \qquad (4.19)$$

$$\theta E_+ = E_- \theta E_+,\, \theta E_- = E_+ \theta E_-\}.$$

Fix $\theta \in Ref\,(\mathscr{H})$, so that $\theta^2 = I_{\mathscr{H}}$, $\theta^* = \theta$, set:

$$\mathscr{E}\,(\theta) := \{(E_0, E_\pm)\,;\, \theta E_0 = E_0,\, \theta E_+ = E_- \theta E_+,\, \theta E_- = E_+ \theta E_-\}.$$
$$(4.20)$$

Question. Let $\varepsilon = (E_0, E_\pm)$ be given, and suppose $E_+ \theta E_+ \geq 0$ for <u>all</u> $\theta \in Ref(\mathscr{H})$, then does it follow that $E_+ E_0 E_- = E_+ E_-$ holds?

Theorem 4.5. Given an infinite-dimensional complex Hilbert space $\mathscr{H}$, let the setting be as above, i.e., reflections, Markov property, and O.S.-positivity defined as stated. Then

$$\bigcap_{\theta \in Ref(\mathscr{H})} \mathscr{E}_{OS}(\theta) = \mathscr{E}(Markov). \tag{4.21}$$

Markov processes and Markov reflection positivity

The axioms for the system are as follows:

1. $\theta E_0 = E_0$;
2. $E_+ \theta E_- = \theta E_-$;
3. $E_- \theta E_+ = \theta E_+$;
4. the O.S.-positivity holds, i.e.,

$$E_+ \theta E_+ \geq 0; \qquad (4.22)$$

5. $\theta U \theta = U^*$, or $\theta U(g) \theta = U(g^{-1})$.

Setting

It is further assumed that, for some subgroup $S \subset G$, we have $U(s) \mathscr{H}_+ \subset \mathscr{H}_+$, $\forall s \in S$; or equivalently,

$$E_+ U(s) E_+ = U(s) E_+, \ s \in S. \qquad (4.23)$$

It is shown above that the additional Markov-restriction

$$E_+ E_0 E_- = E_+ E_- \qquad (4.24)$$

is "very" strong. Moreover, if θ is fixed, then $(4.24) \implies (4.22)$.
Proof Sketch: Assume (4.24), then

$$
\begin{aligned}
E_+ \theta E_+ &= E_+ E_- \theta E_+ \\
&= E_+ E_0 E_- \theta E_+ = \underbrace{E_+ E_0 \theta E_+}_{=E_0} \\
&= E_+ E_0 E_+ \geq 0,
\end{aligned}
$$

so (4.22) holds (with additional information about the OS operator $E_+ \theta E_+$.)

Covariance operator

- V: a LCTVS, locally convex topological vector space.
- G: a Lie group.
- U: a unitary representation of G.
- Let $\{\psi_{v,g}\}_{(v,g) \in V \times G}$ be a real valued stochastic process s.t. $\psi_{v,g} \in \mathscr{H} = L^2(\Omega, \mathscr{F}, \mathbb{P})$, and

$$\mathbb{E}(\psi_{v,g}) = 0, \ (v, g) \in V \times G. \qquad (4.25)$$

- Further assume that a reflection θ is given, and that

$$\theta(\psi_{v,g}) = \psi_{v,g^{-1}}, \ (v, g) \in V \times G. \qquad (4.26)$$

Definition 4.6. Let (v_i, g_i), $i = 1, 2$, be given, and set

$$\mathbb{E}\left(\psi_{v_1, g_1} \psi_{v_2, g_2}\right) = \langle v_1, r\left(g_1, g_2\right) v_2 \rangle \qquad (4.27)$$

where $\langle \cdot, \cdot \rangle$ is a fixed positive definite Hermitian inner product on V.

Hence (4.27) determines a function r on $G \times G$; it is operator valued, taking values in operators in V. This function is called the *covariance operator*.

Two specializations:

1. $G = \mathbb{R}$, $S = \mathbb{R}_+ \cup \{0\} = [0, \infty)$, and

2. the process is stationary:

$$\mathbb{E}\left(\psi_{v_1, t_1} \psi_{v_2, t_2}\right) = \langle v_1, r\left(t_1 - t_2\right) v_2 \rangle, \qquad (4.28)$$

$\forall t_1, t_2 \in \mathbb{R}$, $\forall v_1, v_2 \in V$.

Theorem 4.7 (A. Klein [Kle77]). Let the stationary stochastic process $\{\psi_{v,t}\}$, $(v,t) \in V \times \mathbb{R}$, be as specified above, and let $\{r(t)\}_{t \in \mathbb{R}}$ be the covariance operator. Set $\theta(\psi_{v,t}) := \psi_{v,-t}$, $t \in \mathbb{R}$. Assume $\langle \psi_+, \theta\psi_+ \rangle \geq 0$, $\forall \psi_+ \in \mathscr{H}_+$, then for $\forall n \in \mathbb{N}$, $\forall \{v_i\}_{i=1}^n \subset V$, $\forall \{t_i\}_{i=1}^n \subset \mathbb{R}_+ \cup \{0\}$, we have

$$\sum_i \sum_j \langle v_i, r(t_i + t_j) v_j \rangle \geq 0; \qquad (4.29)$$

which is the O.S.-positivity condition.

Theorem 4.8 (A. Klein, continued). Moreover, the Markov property $E_+E_0E_- = E_+E_-$ holds iff $r(\cdot)$ is a semigroup, i.e., $r(t+s) = r(t)r(s)$, for $\forall s, t \in [0, \infty)$. In particular, in the case of stationary processes, when O.S.-positivity is assumed, then two conditions hold:

1. the covariance function $r(\cdot)$ is positive definite:

$$\sum_i \sum_j \langle v_i, r(t_i - t_j) v_j \rangle \geq 0; \text{ and}$$

2. condition (4.29) holds as well, i.e.,

$$\sum_i \sum_j \langle v_i, r(t_i + t_j) v_j \rangle \geq 0.$$

Remark 4.9. In the scalar case, a list of stationary positive definite, and Gaussian O.S.-positive, covariance functions $\{r(t)\}_{t \in \mathbb{R}}$ includes:

- $e^{-a|t|}$, $a > 0$, fixed;
- $\dfrac{1 - e^{-b|t|}}{b|t|}$, $b > 0$, fixed;
- $\dfrac{1}{1 + |t|}$;
- $\dfrac{1}{\sqrt{1 + |t|}} e^{-\frac{|t|}{1 + |t|}}$, $t \in \mathbb{R}$.

Only $r(t) := e^{-a|t|}$ is also the generator of a Markov system; it is the Ornstein-Uhlenbeck process.

Corollary 4.10. Let $\{\psi_{v,t}\}$ be as specified above, $\theta \, \psi_{v,t} = \psi_{v,-t}$, and assume O.S.-p holds. Let $\mathscr{K}$ denote the Hilbert completion of $span \, \{\psi_{v,t} \, ; \, t \geq 0\}$ with respect to the induced inner product from (4.29). Then a selfadjoint and contractive semigroup $\{R(s) \, ; \, s \geq 0\}$ is well defined by $R(s) \, \psi_{v,t} := \psi_{v,t+s}$; i.e., $\{R(s)\}_{s \geq 0}$ is a selfadjoint contractive semigroup of operators in $\mathscr{K}$, $R(s + s') = R(s) \, R(s')$.

Proof. Note that

$$
\begin{aligned}
\langle R(s) \, \psi_{v_1,t_1}, \psi_{v_2,t_2} \rangle_{\mathscr{K}} &= \langle \psi_{v_1,t_1}, R(s) \, \psi_{v_2,t_2} \rangle_{\mathscr{K}} \\
&= \langle v_1, r(t_1 + t_2 + s) \, v_2 \rangle_V.
\end{aligned}
$$

$\square$

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