Operator algebras on locally compact abelian groups

Raffael Hagger

Kiel University
Christian-Albrechts-Universität zu Kiel

Workshop on Quantum Harmonic Analysis, August 5, 2024

Joint work with Robert Fulsche.

Let G be a locally compact abelian group (such as $\mathbb{Z}$ or $\mathbb{R}$) and consider a bounded integral operator of the form

$$A\colon L^2(G) \to L^2(G), \quad (Af)(x) := \int_G k(x,y) f(y) \, \mathrm{d}y.$$

Questions:

- When is this integral operator compact?
- When is this integral operator Fredholm?
- What is the essential spectrum of A?

Let G be a locally compact abelian group (such as $\mathbb{Z}$ or $\mathbb{R}$) and consider a bounded integral operator of the form

$$A\colon L^2(G) \to L^2(G), \quad (Af)(x) := \int_G k(x,y)f(y)\,\mathrm{d}y.$$

Questions:

- When is this integral operator compact?
- When is this integral operator Fredholm?
- What is the essential spectrum of A?

Let G be a locally compact abelian group (such as $\mathbb{Z}$ or $\mathbb{R}$) and consider a bounded integral operator of the form

$$A\colon L^2(G) \to L^2(G), \quad (Af)(x) := \int_G k(x,y) f(y) \, \mathrm{d}y.$$

Questions:

- When is this integral operator compact?
- When is this integral operator Fredholm?
- What is the essential spectrum of A?

Let G be a locally compact abelian group (such as $\mathbb{Z}$ or $\mathbb{R}$) and consider a bounded integral operator of the form

$$A\colon L^2(G) \to L^2(G), \quad (Af)(x) := \int_G k(x,y)f(y)\,\mathrm{d}y.$$

Questions:

- When is this integral operator compact?
- When is this integral operator Fredholm?
- What is the essential spectrum of A?

- $G := \mathbb{Z}$, i.e. $L^2(G) = \ell^2(\mathbb{Z})$
- $A\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ can be described by

$$(Af)_j = \sum_{k\in\mathbb{Z}} A_{jk} f_k,$$

that is,

$$A = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & A_{-1-1} & A_{-10} & A_{-11} & \cdots \\ \cdots & A_{0-1} & A_{00} & A_{01} & \cdots \\ \cdots & A_{1-1} & A_{10} & A_{11} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

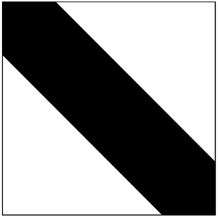
- $G := \mathbb{Z}$, i.e. $L^2(G) = \ell^2(\mathbb{Z})$
- $A\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ can be described by

$$(Af)_j = \sum_{k \in \mathbb{Z}} A_{jk} f_k,$$

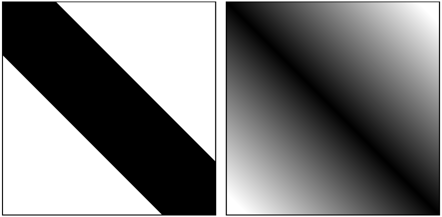
that is,

$$A = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & A_{-1-1} & A_{-10} & A_{-11} & \cdots \\ \cdots & A_{0-1} & A_{00} & A_{01} & \cdots \\ \cdots & A_{1-1} & A_{10} & A_{11} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

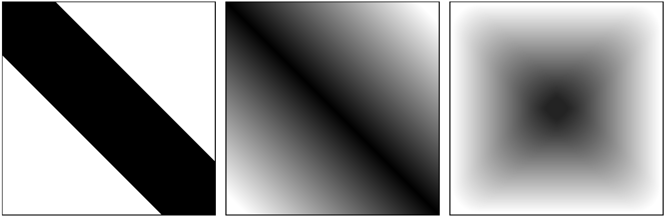
- band operators ($\mathrm{BO}(\mathbb{Z})$): finitely many non-zero diagonals
- band-dominated operators ($\mathrm{BDO}(\mathbb{Z})$): closure of $\mathrm{BO}(\mathbb{Z})$ w.r.t. $\|\cdot\|_{\mathcal{L}(\ell^2(\mathbb{Z}))}$
- $\mathrm{BDO}(\mathbb{Z})$ is a C^*-algebra that contains all compact operators



- band operators $(\mathrm{BO}(\mathbb{Z}))$: finitely many non-zero diagonals
- band-dominated operators $(\mathrm{BDO}(\mathbb{Z}))$: closure of $\mathrm{BO}(\mathbb{Z})$ w.r.t. $\|\cdot\|_{\mathcal{L}(\ell^2(\mathbb{Z}))}$
- $\mathrm{BDO}(\mathbb{Z})$ is a C^*-algebra that contains all compact operators



- band operators ($\mathrm{BO}(\mathbb{Z})$): finitely many non-zero diagonals
- band-dominated operators ($\mathrm{BDO}(\mathbb{Z})$): closure of $\mathrm{BO}(\mathbb{Z})$ w.r.t. $\|\cdot\|_{\mathcal{L}(\ell^2(\mathbb{Z}))}$
- $\mathrm{BDO}(\mathbb{Z})$ is a C^*-algebra that contains all compact operators



$A \in \mathrm{BDO}(\mathbb{Z})$

$$A = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & A_{-2-2} & A_{-2-1} & A_{-20} & A_{-21} & A_{-22} & \dots \\ \dots & A_{-1-2} & A_{-1-1} & A_{-10} & A_{-11} & A_{-12} & \dots \\ \dots & A_{0-2} & A_{0-1} & A_{00} & A_{01} & A_{02} & \dots \\ \dots & A_{1-2} & A_{1-1} & A_{10} & A_{11} & A_{12} & \dots \\ \dots & A_{2-2} & A_{2-1} & A_{20} & A_{21} & A_{22} & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$A \in \mathrm{BDO}(\mathbb{Z})$

$$A + K = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & A_{-2-2} & A_{-2-1} & A_{-20} & A_{-21} & A_{-22} & \dots \\ \dots & A_{-1-2} & A_{-1-1} & A_{-10} & & A_{-12} & \dots \\ \dots & A_{0-2} & A_{0-1} & A_{00} & A_{01} & A_{02} & \dots \\ \dots & A_{1-2} & A_{1-1} & A_{10} & A_{11} & A_{12} & \dots \\ \dots & A_{2-2} & A_{2-1} & A_{20} & A_{21} & A_{22} & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$A \iff A + K$ compact and $\mathrm{sp}_{\mathrm{ess}}(A) = \mathrm{sp}_{\mathrm{ess}}(A + K)$

$A \in \mathrm{BDO}(\mathbb{Z})$

$$A + K = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & A_{-2-2} & A_{-2-1} & A_{-20} & A_{-21} & A_{-22} & \cdots \\ \cdots & A_{-1-2} & A_{-1-1} & A_{-10} & & A_{-12} & \cdots \\ \cdots & A_{0-2} & A_{0-1} & A_{00} & A_{01} & A_{02} & \cdots \\ \cdots & A_{1-2} & A_{1-1} & & A_{11} & A_{12} & \cdots \\ \cdots & A_{2-2} & A_{2-1} & A_{20} & A_{21} & A_{22} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$A \Longleftrightarrow A + K$ compact and $\mathrm{sp}_{\mathrm{ess}}(A) = \mathrm{sp}_{\mathrm{ess}}(A + K)$

$A \in \mathrm{BDO}(\mathbb{Z})$

$$A + K = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & A_{-2-2} & A_{-2-1} & A_{-20} & A_{-21} & A_{-22} & \cdots \\ \cdots & A_{-1-2} & & & & A_{-12} & \cdots \\ \cdots & A_{0-2} & & & & A_{02} & \cdots \\ \cdots & A_{1-2} & & & & A_{12} & \cdots \\ \cdots & A_{2-2} & A_{2-1} & A_{20} & A_{21} & A_{22} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$A \Longleftrightarrow A + K$ compact and $\mathrm{sp}_{\mathrm{ess}}(A) = \mathrm{sp}_{\mathrm{ess}}(A + K)$

$A \in \mathrm{BDO}(\mathbb{Z})$

$$A + K = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & A_{-2-2} & A_{-2-1} & A_{-20} & A_{-21} & A_{-22} & \cdots \\ \cdots & A_{-1-2} & & & & A_{-12} & \cdots \\ \cdots & A_{0-2} & & & & A_{02} & \cdots \\ \cdots & A_{1-2} & & & & A_{12} & \cdots \\ & & A_{2-1} & A_{20} & A_{21} & A_{22} & \cdots \\ & & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$A \Longleftrightarrow A + K$ compact and $\mathrm{sp}_{\mathrm{ess}}(A) = \mathrm{sp}_{\mathrm{ess}}(A + K)$

$A \in \mathrm{BDO}(\mathbb{Z})$

$$A = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & A_{-2-2} & A_{-2-1} & A_{-20} & A_{-21} & A_{-22} & \cdots \\ \cdots & A_{-1-2} & A_{-1-1} & A_{-10} & A_{-11} & A_{-12} & \cdots \\ \cdots & A_{0-2} & A_{0-1} & \boxed{A_{00}} & A_{01} & A_{02} & \cdots \\ \cdots & A_{1-2} & A_{1-1} & A_{10} & A_{11} & A_{12} & \cdots \\ \cdots & A_{2-2} & A_{2-1} & A_{20} & A_{21} & A_{22} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$A \in \mathrm{BDO}(\mathbb{Z})$

$$V^{-1}AV = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & A_{-1-1} & A_{-10} & A_{-11} & A_{-12} & A_{-13} & \cdots \\ \cdots & A_{0-1} & \boxed{A_{00}} & A_{01} & A_{02} & A_{03} & \cdots \\ \cdots & A_{1-1} & A_{10} & A_{11} & A_{12} & A_{13} & \cdots \\ \cdots & A_{2-1} & A_{20} & A_{21} & A_{22} & A_{23} & \cdots \\ \cdots & A_{3-1} & A_{30} & A_{31} & A_{32} & A_{33} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$A \in \mathrm{BDO}(\mathbb{Z})$

$$V^{-2}AV^2 = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & \boxed{A_{00}} & A_{01} & A_{02} & A_{03} & A_{04} & \cdots \\ \cdots & A_{10} & A_{11} & A_{12} & A_{13} & A_{14} & \cdots \\ \cdots & A_{20} & A_{21} & A_{22} & A_{23} & A_{24} & \cdots \\ \cdots & A_{30} & A_{31} & A_{32} & A_{33} & A_{34} & \cdots \\ \cdots & A_{40} & A_{41} & A_{42} & A_{43} & A_{44} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$A \in \mathrm{BDO}(\mathbb{Z})$

$$V^{-5}AV^{5} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} & \cdots \\ \cdots & A_{43} & A_{44} & A_{45} & A_{46} & A_{47} & \cdots \\ \cdots & A_{53} & A_{54} & A_{55} & A_{56} & A_{57} & \cdots \\ \cdots & A_{63} & A_{64} & A_{65} & A_{66} & A_{67} & \cdots \\ \cdots & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$A \in \mathrm{BDO}(\mathbb{Z})$

$$B = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & B_{-2-2} & B_{-2-1} & B_{-20} & B_{-21} & B_{-22} & \dots \\ \dots & B_{-1-2} & B_{-1-1} & B_{-10} & B_{-11} & B_{-12} & \dots \\ \dots & B_{0-2} & B_{0-1} & B_{00} & B_{01} & B_{02} & \dots \\ \dots & B_{1-2} & B_{1-1} & B_{10} & B_{11} & B_{12} & \dots \\ \dots & B_{2-2} & B_{2-1} & B_{20} & B_{21} & B_{22} & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$A_h f := Bf = \lim_{n \to \infty} V^{-h_n} A V^{h_n} f$ (if it exists for all $f \in \ell^2(\mathbb{Z})$)

$\alpha.(A) : \mathbb{Z} \to \mathcal{L}(\ell^2(\mathbb{Z}))$, $\alpha_n(A) := V^{-n}AV^n$ can be extended to a strongly continuous map on $\beta\mathbb{Z}$.

Theorem

$A\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is compact if and only if $A \in \mathrm{BDO}(\mathbb{Z})$ and $\alpha_x(A) = 0$ for all $x \in \beta\mathbb{Z} \setminus \mathbb{Z}$.

Theorem (Lindner/Seidel 2014, Lange/Rabinovich 1985)

Let $A \in \mathrm{BDO}(\mathbb{Z})$. A is Fredholm if and only all of its limit operators $\alpha_x(A)$, $x \in \beta\mathbb{Z} \setminus \mathbb{Z}$, are invertible. Moreover,

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{x \in \beta\mathbb{Z}\setminus\mathbb{Z}} \mathrm{sp}(\alpha_x(A)).$$

$\alpha.(A) : \mathbb{Z} \to \mathcal{L}(\ell^2(\mathbb{Z}))$, $\alpha_n(A) := V^{-n}AV^n$ can be extended to a strongly continuous map on $\beta\mathbb{Z}$.

Theorem

$A\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is compact if and only if $A \in \mathrm{BDO}(\mathbb{Z})$ and $\alpha_x(A) = 0$ for all $x \in \beta\mathbb{Z} \setminus \mathbb{Z}$.

Theorem (Lindner/Seidel 2014, Lange/Rabinovich 1985)

Let $A \in \mathrm{BDO}(\mathbb{Z})$. A is Fredholm if and only all of its limit operators $\alpha_x(A)$, $x \in \beta\mathbb{Z} \setminus \mathbb{Z}$, are invertible. Moreover,

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{x \in \beta\mathbb{Z} \setminus \mathbb{Z}} \mathrm{sp}(\alpha_x(A)).$$

$\alpha_.(A) : \mathbb{Z} \to \mathcal{L}(\ell^2(\mathbb{Z}))$, $\alpha_n(A) := V^{-n}AV^n$ can be extended to a strongly continuous map on $\beta\mathbb{Z}$.

Theorem

$A\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is compact if and only if $A \in \mathrm{BDO}(\mathbb{Z})$ and $\alpha_x(A) = 0$ for all $x \in \beta\mathbb{Z} \setminus \mathbb{Z}$.

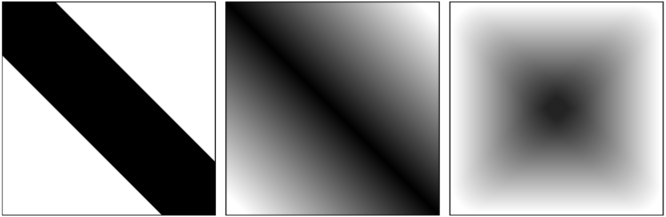
Theorem (Lindner/Seidel 2014, Lange/Rabinovich 1985)

Let $A \in \mathrm{BDO}(\mathbb{Z})$. A is Fredholm if and only all of its limit operators $\alpha_x(A)$, $x \in \beta\mathbb{Z} \setminus \mathbb{Z}$, are invertible. Moreover,

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{x \in \beta\mathbb{Z}\setminus\mathbb{Z}} \mathrm{sp}(\alpha_x(A)).$$

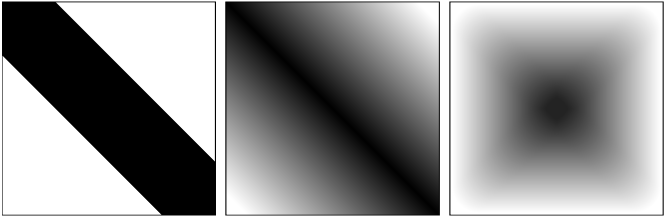
Now consider $G := \mathbb{R}$ and recall $Af(x) := \int_{\mathbb{R}} k(x,y)f(y)\,\mathrm{d}y$.

- band operators ($\mathrm{BO}(\mathbb{Z})$): finitely many non-zero diagonals
- band-dominated operators ($\mathrm{BDO}(\mathbb{Z})$): closure of $\mathrm{BO}(\mathbb{Z})$ w.r.t. $\|\cdot\|_{\mathcal{L}(\ell^2(\mathbb{Z}))}$
- $\mathrm{BDO}(\mathbb{Z})$ is a C^*-algebra that contains all compact operators



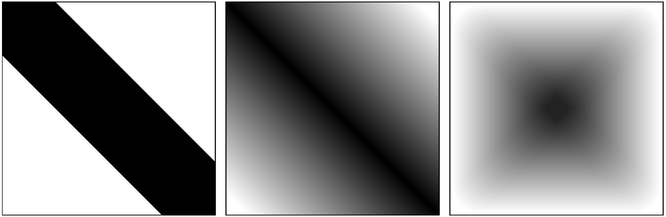
Now consider $G := \mathbb{R}$ and recall $Af(x) := \int_{\mathbb{R}} k(x,y)f(y)\,\mathrm{d}y$.

- band operators ($\mathrm{BO}(\mathbb{Z})$): finitely many non-zero diagonals
- band-dominated operators ($\mathrm{BDO}(\mathbb{Z})$): closure of $\mathrm{BO}(\mathbb{Z})$ w.r.t. $\|\cdot\|_{\mathcal{L}(\ell^2(\mathbb{Z}))}$
- $\mathrm{BDO}(\mathbb{Z})$ is a C^*-algebra that contains all compact operators



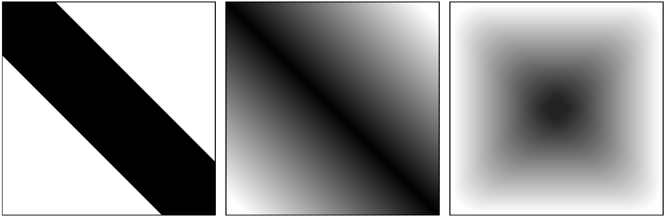
Now consider $G := \mathbb{R}$ and recall $Af(x) := \int_{\mathbb{R}} k(x,y)f(y)\,\mathrm{d}y$.

- band operators ($\mathrm{BO}(\mathbb{R})$): $k(x,y) = 0$ for $|x-y| > \omega$
- band-dominated operators ($\mathrm{BDO}(\mathbb{R})$): closure of $\mathrm{BO}(\mathbb{R})$ w.r.t. $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}))}$
- $\mathrm{BDO}(\mathbb{R})$ is a C^*-algebra that contains all compact operators



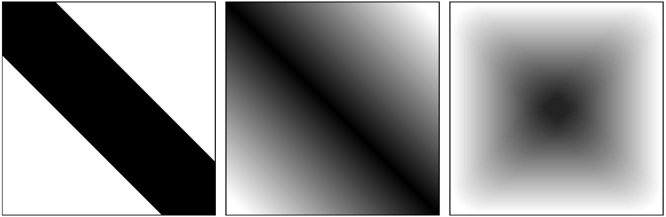
Now consider $G := \mathbb{R}$ and recall $Af(x) := \int_{\mathbb{R}} k(x,y)f(y)\,\mathrm{d}y$.

- band operators ($\mathrm{BO}(\mathbb{R})$): $k(x,y) = 0$ for $|x-y| > \omega$
- band-dominated operators ($\mathrm{BDO}(\mathbb{R})$): closure of $\mathrm{BO}(\mathbb{R})$ w.r.t. $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}))}$
- $\mathrm{BDO}(\mathbb{R})$ is a C^*-algebra that contains all compact operators



Now consider $G := \mathbb{R}$ and recall $Af(x) := \int_{\mathbb{R}} k(x,y)f(y)\,\mathrm{d}y$.

- band operators $(\mathrm{BO}(\mathbb{R}))$: $k(x,y) = 0$ for $|x-y| > \omega$
- band-dominated operators $(\mathrm{BDO}(\mathbb{R}))$: closure of $\mathrm{BO}(\mathbb{R})$ w.r.t. $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}))}$
- $\mathrm{BDO}(\mathbb{R})$ is a C^*-algebra that contains all compact operators



- Problem 1: A is not necessarily compact if k has compact support.
- Problem 2: Even for multiplication operators, limit operators may not exist, e.g.,

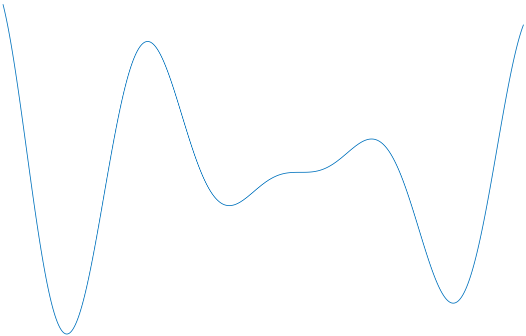
$$f(x) := (-1)^{\lfloor x \rfloor}.$$

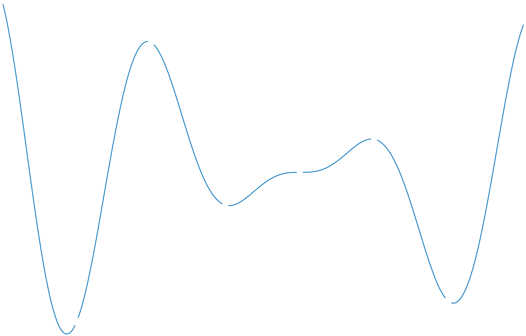
Then $(V_{-\sqrt{2}n} M_f V_{\sqrt{2}n})_{n \in \mathbb{N}}$ does not have a strongly convergent subsequence.

- Problem 1: A is not necessarily compact if k has compact support.
- Problem 2: Even for multiplication operators, limit operators may not exist, e.g.,

$$f(x) := (-1)^{\lfloor x \rfloor}.$$

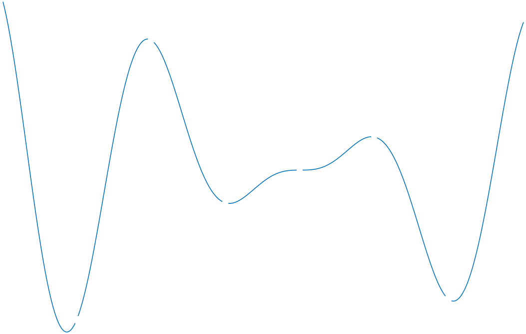
Then $(V_{-\sqrt{2}n} M_f V_{\sqrt{2}n})_{n \in \mathbb{N}}$ does not have a strongly convergent subsequence.





$$L^2(\mathbb{R}) \cong \ell^2(\mathbb{Z}, L^2([0,1)))$$

$\Longrightarrow$ $A \in \mathrm{BDO}(\mathbb{R})$ is invertible modulo $\mathcal{K}(L^2(\mathbb{R}), \mathcal{P})$ if and only if all limit operators of A are invertible.*



$$L^2(\mathbb{R}) \cong \ell^2(\mathbb{Z}, L^2([0,1)))$$

$\Longrightarrow A \in \mathrm{BDO}(\mathbb{R})$ is invertible modulo $\mathcal{K}(L^2(\mathbb{R}), \mathcal{P})$ if and only if all limit operators of A are invertible.*

Let Ξ be a locally compact abelian group and

$$U_{\cdot} : \Xi \to \mathcal{L}(\mathcal{H}), \quad \xi \mapsto U_\xi$$

a strongly continuous, unitary, projective representation, i.e.,

$$U_\xi U_\rho = m(\xi, \rho) U_{\xi\rho}.$$

For $\Xi := G \times \widehat{G}$ we may choose $\mathcal{H} = L^2(G)$ and

$$U_{g,\chi} f(h) = \chi(h) f(g^{-1}h).$$

For example, if $\Xi = \mathbb{Z} \times \mathbb{T}$, then

$$U_{n,t} f(m) = t^m f(m-n)$$

for $f \in \ell^2(\mathbb{Z})$, $m, n \in \mathbb{Z}$, $t \in \mathbb{T}$.

Let Ξ be a locally compact abelian group and

$$U_\cdot : \Xi \to \mathcal{L}(\mathcal{H}), \quad \xi \mapsto U_\xi$$

a strongly continuous, unitary, projective representation, i.e.,

$$U_\xi U_\rho = m(\xi, \rho) U_{\xi\rho}.$$

For $\Xi := G \times \widehat{G}$ we may choose $\mathcal{H} = L^2(G)$ and

$$U_{g,\chi} f(h) = \chi(h) f(g^{-1}h).$$

For example, if $\Xi = \mathbb{Z} \times \mathbb{T}$, then

$$U_{n,t} f(m) = t^m f(m-n)$$

for $f \in \ell^2(\mathbb{Z})$, $m, n \in \mathbb{Z}$, $t \in \mathbb{T}$.

Let Ξ be a locally compact abelian group and

$$U.: \Xi \to \mathcal{L}(\mathcal{H}), \quad \xi \mapsto U_\xi$$

a strongly continuous, unitary, projective representation, i.e.,

$$U_\xi U_\rho = m(\xi, \rho) U_{\xi\rho}.$$

For $\Xi := G \times \widehat{G}$ we may choose $\mathcal{H} = L^2(G)$ and

$$U_{g,\chi} f(h) = \chi(h) f(g^{-1}h).$$

For example, if $\Xi = \mathbb{Z} \times \mathbb{T}$, then

$$U_{n,t} f(m) = t^m f(m-n)$$

for $f \in \ell^2(\mathbb{Z})$, $m, n \in \mathbb{Z}$, $t \in \mathbb{T}$.

Let Ξ be a locally compact abelian group and

$$U_\cdot : \Xi \to \mathcal{L}(\mathcal{H}), \quad \xi \mapsto U_\xi$$

a strongly continuous, unitary, projective representation, i.e.,

$$U_\xi U_\rho = m(\xi, \rho) U_{\xi\rho}.$$

For $\Xi := G \times \widehat{G}$ we may choose $\mathcal{H} = L^2(G)$ and

$$U_{g,\chi} f(h) = \chi(h) f(g^{-1}h).$$

For example, if $\Xi = \mathbb{R} \times \mathbb{R}$, then

$$U_{x,t} f(y) = e^{iyt} f(y - x)$$

for $f \in L^2(\mathbb{R})$, $x, y, t \in \mathbb{R}$.

Let $\mathcal{C}_1(G) := \{A \in \mathcal{L}(L^2(G)) : z \mapsto \alpha_z(A)$ continuous$\}$.

Proposition

Let $A \in \mathcal{C}_1(G)$. Then the map

$$\alpha_\cdot : G \times \widehat{G} \to \mathcal{L}(L^2(G)), \quad \alpha_z(A) := U_z A U_z^*$$

extends to a strongly continuous map on $\mathcal{M}$.

Here $\mathcal{M}$ denotes the maximal ideal space of $\mathrm{BUC}(G \times \widehat{G})$, which was defined in Robert's talk.

Theorem (Fulsche/H. '24)

Let $A \in \mathcal{C}_1(G)$. A is Fredholm if and only if $\alpha_z(A)$ is invertible for every $z \in \mathcal{M} \setminus (G \times \widehat{G})$ (and the inverses are uniformly bdd).

Let $\mathcal{C}_1(G) := \{A \in \mathcal{L}(L^2(G)) : z \mapsto \alpha_z(A) \text{ continuous}\}$.

Proposition

Let $A \in \mathcal{C}_1(G)$. Then the map

$$\alpha_\cdot : G \times \widehat{G} \to \mathcal{L}(L^2(G)), \quad \alpha_z(A) := U_z A U_z^*$$

extends to a strongly continuous map on $\mathcal{M}$.

Here $\mathcal{M}$ denotes the maximal ideal space of $\mathrm{BUC}(G \times \widehat{G})$, which was defined in Robert's talk.

Theorem (Fulsche/H. '24)

Let $A \in \mathcal{C}_1(G)$. A is Fredholm if and only if $\alpha_z(A)$ is invertible for every $z \in \mathcal{M} \setminus (G \times \widehat{G})$ (and the inverses are uniformly bdd).

Let $\mathcal{C}_1(G) := \{A \in \mathcal{L}(L^2(G)) : z \mapsto \alpha_z(A)$ continuous$\}$.

Proposition

Let $A \in \mathcal{C}_1(G)$. Then the map

$$\alpha_\cdot : G \times \widehat{G} \to \mathcal{L}(L^2(G)), \quad \alpha_z(A) := U_z A U_z^*$$

extends to a strongly continuous map on $\mathcal{M}$.

Here $\mathcal{M}$ denotes the maximal ideal space of $\mathrm{BUC}(G \times \widehat{G})$, which was defined in Robert's talk.

Theorem (Fulsche/H. '24)

Let $A \in \mathcal{C}_1(G)$. A is Fredholm if and only if $\alpha_z(A)$ is invertible for every $z \in \mathcal{M} \setminus (G \times \widehat{G})$ (and the inverses are uniformly bdd).

Theorem (Fulsche/H. '24)

Let $A \in \mathcal{C}_1(G)$. A is Fredholm if and only if $\alpha_z(A)$ is invertible for every $z \in \mathcal{M} \setminus (G \times \widehat{G})$.

Theorem (Lindner/Seidel 2014, Lange/Rabinovich 1985)

Let $A \in \mathrm{BDO}(\mathbb{Z})$. A is Fredholm if and only all of its limit operators $\alpha_x(A)$, $x \in \beta\mathbb{Z} \setminus \mathbb{Z}$, are invertible.

Observations:

- $\mathcal{C}_1(\mathbb{Z}) = \mathrm{BDO}(\mathbb{Z})$ (well known, see e.g. the book by Rabinovich, Roch, Silbermann)
- $\mathcal{M} \cong \beta\mathbb{Z} \times \mathbb{T}$ in this case, hence $\mathcal{M} \setminus (\mathbb{Z} \times \mathbb{T}) \cong (\beta\mathbb{Z} \setminus \mathbb{Z}) \times \mathbb{T}$
- $\alpha_{x,t}(A)$ is unitarily equivalent to $\alpha_{x,1}(A) \equiv \alpha_x(A)$ for all $x \in \beta\mathbb{Z}$, $t \in \mathbb{T}$

Theorem (Fulsche/H. '24)

Let $A \in \mathcal{C}_1(G)$. A is Fredholm if and only if $\alpha_z(A)$ is invertible for every $z \in \mathcal{M} \setminus (G \times \widehat{G})$.

Theorem (Lindner/Seidel 2014, Lange/Rabinovich 1985)

Let $A \in \mathrm{BDO}(\mathbb{Z})$. A is Fredholm if and only all of its limit operators $\alpha_x(A)$, $x \in \beta\mathbb{Z} \setminus \mathbb{Z}$, are invertible.

Observations:

- $\mathcal{C}_1(\mathbb{Z}) = \mathrm{BDO}(\mathbb{Z})$ (well known, see e.g. the book by Rabinovich, Roch, Silbermann)
- $\mathcal{M} \cong \beta\mathbb{Z} \times \mathbb{T}$ in this case, hence $\mathcal{M} \setminus (\mathbb{Z} \times \mathbb{T}) \cong (\beta\mathbb{Z} \setminus \mathbb{Z}) \times \mathbb{T}$
- $\alpha_{x,t}(A)$ is unitarily equivalent to $\alpha_{x,1}(A) \equiv \alpha_x(A)$ for all $x \in \beta\mathbb{Z}$, $t \in \mathbb{T}$

Theorem (Fulsche/H. '24)

Let $A \in \mathcal{C}_1(G)$. A is Fredholm if and only if $\alpha_z(A)$ is invertible for every $z \in \mathcal{M} \setminus (G \times \widehat{G})$.

Theorem (Lindner/Seidel 2014, Lange/Rabinovich 1985)

Let $A \in \mathrm{BDO}(\mathbb{Z})$. A is Fredholm if and only all of its limit operators $\alpha_x(A)$, $x \in \beta\mathbb{Z} \setminus \mathbb{Z}$, are invertible.

Observations:

- $\mathcal{C}_1(\mathbb{Z}) = \mathrm{BDO}(\mathbb{Z})$ (well known, see e.g. the book by Rabinovich, Roch, Silbermann)
- $\mathcal{M} \cong \beta\mathbb{Z} \times \mathbb{T}$ in this case, hence $\mathcal{M} \setminus (\mathbb{Z} \times \mathbb{T}) \cong (\beta\mathbb{Z} \setminus \mathbb{Z}) \times \mathbb{T}$
- $\alpha_{x,t}(A)$ is unitarily equivalent to $\alpha_{x,1}(A) \equiv \alpha_x(A)$ for all $x \in \beta\mathbb{Z}$, $t \in \mathbb{T}$

Theorem (Fulsche/H. '24)

Let $A \in \mathcal{C}_1(G)$. A is Fredholm if and only if $\alpha_z(A)$ is invertible for every $z \in \mathcal{M} \setminus (G \times \widehat{G})$.

Theorem (Lindner/Seidel 2014, Lange/Rabinovich 1985)

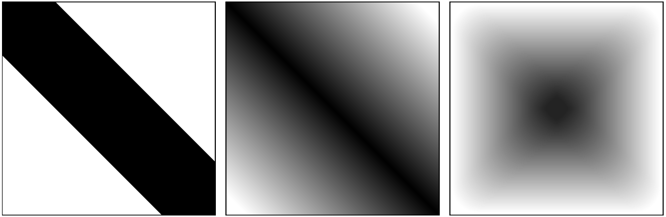
Let $A \in \mathrm{BDO}(\mathbb{Z})$. A is Fredholm if and only all of its limit operators $\alpha_x(A)$, $x \in \beta\mathbb{Z} \setminus \mathbb{Z}$, are invertible.

Observations:

- $\mathcal{C}_1(\mathbb{Z}) = \mathrm{BDO}(\mathbb{Z})$ (well known, see e.g. the book by Rabinovich, Roch, Silbermann)
- $\mathcal{M} \cong \beta\mathbb{Z} \times \mathbb{T}$ in this case, hence $\mathcal{M} \setminus (\mathbb{Z} \times \mathbb{T}) \cong (\beta\mathbb{Z} \setminus \mathbb{Z}) \times \mathbb{T}$
- $\alpha_{x,t}(A)$ is unitarily equivalent to $\alpha_{x,1}(A) \equiv \alpha_x(A)$ for all $x \in \beta\mathbb{Z}$, $t \in \mathbb{T}$

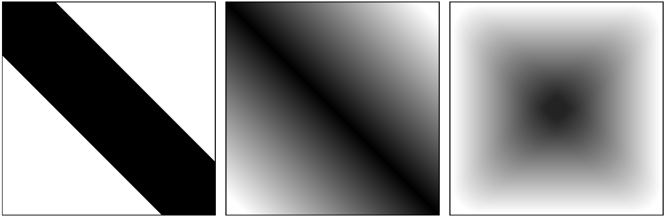
Now consider $G := \mathbb{R}$ and recall $Af(x) := \int_{\mathbb{R}} k(x,y)f(y)\,\mathrm{d}y$.

- band operators $(\mathrm{BO}(\mathbb{R}))$: $k(x,y) = 0$ for $|x-y| > \omega$
- band-dominated operators $(\mathrm{BDO}(\mathbb{R}))$: closure of $\mathrm{BO}(\mathbb{R})$ w.r.t. $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}))}$
- $\mathrm{BDO}(\mathbb{R})$ is a C^*-algebra that contains all compact operators



Now let G be an lca group, recall $Af(x) := \int_G k(x,y)f(y)\,\mathrm{d}y$.

- band operators $(\mathrm{BO}(G))$: $k(x,y) = 0$ for $d(x,y) > \omega$
- band-dominated operators $(\mathrm{BDO}(G))$: closure of $\mathrm{BO}(G)$ w.r.t. $\|\cdot\|_{\mathcal{L}(L^2(G))}$
- $\mathrm{BDO}(G)$ is a C^*-algebra that contains all compact operators



Definition

Let G be an lca group. An operator $A \in \mathcal{L}(L^2(G))$ is called a band operator if there is a compact set $K \subseteq G$ such that for all $H \subseteq G$ and all $f \in L^2(G)$ with $\operatorname{supp} f \subseteq H$ we have

$$\operatorname{supp}(Af) \subseteq HK.$$

- equivalently, if $Af(x) := \int_G k(x,y)f(y)\,\mathrm{d}y$, then

$$\operatorname{supp} k \subseteq \{(x, x+y) : x \in G, y \in K\}.$$

- band-dominated operators ($\mathrm{BDO}(G)$): closure of $\mathrm{BO}(G)$ w.r.t. $\|\cdot\|_{\mathcal{L}(L^2(G))}$
- $\mathrm{BDO}(G)$ is a C^*-algebra that contains all compact operators
- If G admits a proper metric (i.e. is second countable), then this agrees with the metric definition.

Definition

Let G be an lca group. An operator $A \in \mathcal{L}(L^2(G))$ is called a band operator if there is a compact set $K \subseteq G$ such that for all $H \subseteq G$ and all $f \in L^2(G)$ with $\operatorname{supp} f \subseteq H$ we have

$$\operatorname{supp}(Af) \subseteq HK.$$

- equivalently, if $Af(x) := \int_G k(x,y)f(y)\,\mathrm{d}y$, then

$$\operatorname{supp} k \subseteq \{(x, x+y) : x \in G, y \in K\}.$$

- band-dominated operators ($\mathrm{BDO}(G)$): closure of $\mathrm{BO}(G)$ w.r.t. $\|\cdot\|_{\mathcal{L}(L^2(G))}$
- $\mathrm{BDO}(G)$ is a C^*-algebra that contains all compact operators
- If G admits a proper metric (i.e. is second countable), then this agrees with the metric definition.

Definition

Let G be an lca group. An operator $A \in \mathcal{L}(L^2(G))$ is called a band operator if there is a compact set $K \subseteq G$ such that for all $H \subseteq G$ and all $f \in L^2(G)$ with $\operatorname{supp} f \subseteq H$ we have

$$\operatorname{supp}(Af) \subseteq HK.$$

- equivalently, if $Af(x) := \int_G k(x,y) f(y) \, \mathrm{d}y$, then

$$\operatorname{supp} k \subseteq \{(x, x+y) : x \in G, y \in K\}.$$

- band-dominated operators ($\mathrm{BDO}(G)$): closure of $\mathrm{BO}(G)$ w.r.t. $\|\cdot\|_{\mathcal{L}(L^2(G))}$
- $\mathrm{BDO}(G)$ is a C^*-algebra that contains all compact operators
- If G admits a proper metric (i.e. is second countable), then this agrees with the metric definition.

Definition

Let G be an lca group. An operator $A \in \mathcal{L}(L^2(G))$ is called a band operator if there is a compact set $K \subseteq G$ such that for all $H \subseteq G$ and all $f \in L^2(G)$ with $\operatorname{supp} f \subseteq H$ we have

$$\operatorname{supp}(Af) \subseteq HK.$$

- equivalently, if $Af(x) := \int_G k(x,y)f(y)\,\mathrm{d}y$, then
$$\operatorname{supp} k \subseteq \{(x, x+y) : x \in G, y \in K\}.$$
- band-dominated operators $(\mathrm{BDO}(G))$: closure of $\mathrm{BO}(G)$ w.r.t. $\|\cdot\|_{\mathcal{L}(L^2(G))}$
- $\mathrm{BDO}(G)$ is a C^*-algebra that contains all compact operators
- If G admits a proper metric (i.e. is second countable), then this agrees with the metric definition.

Definition

Let G be an lca group. An operator $A \in \mathcal{L}(L^2(G))$ is called a band operator if there is a compact set $K \subseteq G$ such that for all $H \subseteq G$ and all $f \in L^2(G)$ with $\operatorname{supp} f \subseteq H$ we have

$$\operatorname{supp}(Af) \subseteq HK.$$

- equivalently, if $Af(x) := \int_G k(x,y)f(y)\,\mathrm{d}y$, then
$$\operatorname{supp} k \subseteq \{(x, x+y) : x \in G, y \in K\}.$$
- band-dominated operators ($\mathrm{BDO}(G)$): closure of $\mathrm{BO}(G)$ w.r.t. $\|\cdot\|_{\mathcal{L}(L^2(G))}$
- $\mathrm{BDO}(G)$ is a C^*-algebra that contains all compact operators
- If G admits a proper metric (i.e. is second countable), then this agrees with the metric definition.

- Problem 1: A is not necessarily compact if k has compact support.
- Problem 2: Even for multiplication operators, limit operators may not exist, e.g.,

$$f(x) := (-1)^{\lfloor x \rfloor}.$$

Then $(V_{-\sqrt{2}n} M_f V_{\sqrt{2}n})_{n \in \mathbb{N}}$ does not have a strongly convergent subsequence.
- Consequence: $\mathrm{BDO}(G) \neq \mathcal{C}_1(G)$ unless G is discrete

- Problem 1: A is not necessarily compact if k has compact support.
- Problem 2: Even for multiplication operators, limit operators may not exist, e.g.,

$$f(x) := (-1)^{\lfloor x \rfloor}.$$

  Then $(V_{-\sqrt{2}n} M_f V_{\sqrt{2}n})_{n \in \mathbb{N}}$ does not have a strongly convergent subsequence.
- Consequence: $\mathrm{BDO}(G) \neq \mathcal{C}_1(G)$ unless G is discrete

$$\mathcal{C}_{0,1}(G) := \{A \in \mathcal{L}(L^2(G)) : \chi \mapsto \alpha_{e,\chi}(A) \text{ continuous}\}$$

Theorem (Fulsche/H. '24)

Let G be an lca group. Then $\mathrm{BDO}(G) = \mathcal{C}_{0,1}(G)$.

$$\mathcal{C}_{1,0}(G) := \{A \in \mathcal{L}(L^2(G)) : g \mapsto \alpha_{g,1}(A) \text{ continuous}\}$$

Corollary

For every lca group G we have

$$\mathcal{C}_1(G) = \mathrm{BDO}(G) \cap \mathcal{F}^{-1}\,\mathrm{BDO}(\widehat{G})\mathcal{F}.$$

$$\mathcal{C}_{0,1}(G) := \{A \in \mathcal{L}(L^2(G)) : \chi \mapsto \alpha_{e,\chi}(A) \text{ continuous}\}$$

Theorem (Fulsche/H. '24)

Let G be an lca group. Then $\mathrm{BDO}(G) = \mathcal{C}_{0,1}(G)$.

$$\mathcal{C}_{1,0}(G) := \{A \in \mathcal{L}(L^2(G)) : g \mapsto \alpha_{g,1}(A) \text{ continuous}\}$$

Corollary

For every lca group G we have

$$\mathcal{C}_1(G) = \mathrm{BDO}(G) \cap \mathcal{F}^{-1}\,\mathrm{BDO}(\widehat{G})\mathcal{F}.$$

$$\mathcal{C}_{0,1}(G) := \{A \in \mathcal{L}(L^2(G)) : \chi \mapsto \alpha_{e,\chi}(A) \text{ continuous}\}$$

Theorem (Fulsche/H. '24)

Let G be an lca group. Then $\mathrm{BDO}(G) = \mathcal{C}_{0,1}(G)$.

$$\mathcal{C}_{1,0}(G) := \{A \in \mathcal{L}(L^2(G)) : g \mapsto \alpha_{g,1}(A) \text{ continuous}\}$$

Corollary

For every lca group G we have

$$\mathcal{C}_1(G) = \mathrm{BDO}(G) \cap \mathcal{F}^{-1}\,\mathrm{BDO}(\widehat{G})\mathcal{F}.$$

$$\mathcal{C}_{0,1}(G) := \left\{A \in \mathcal{L}(L^2(G)) : \chi \mapsto \alpha_{e,\chi}(A) \text{ continuous}\right\}$$

Theorem (Fulsche/H. '24)

Let G be an lca group. Then $\mathrm{BDO}(G) = \mathcal{C}_{0,1}(G)$.

$$\mathcal{C}_{1,0}(G) := \left\{A \in \mathcal{L}(L^2(G)) : g \mapsto \alpha_{g,1}(A) \text{ continuous}\right\}$$

Corollary

For every lca group G we have

$$\mathcal{C}_1(G) = \mathrm{BDO}(G) \cap \mathcal{F}^{-1}\,\mathrm{BDO}(\widehat{G})\mathcal{F}.$$

Theorem (Fulsche/H. '24)

Let G be an lca group. Then $\mathrm{BDO}(G) = \mathcal{C}_{0,1}(G)$.

Sketch of the proof:

We know that

$$\mathcal{C}_1(G) = L^1(G \times \widehat{G}) * \mathcal{L}(L^2(G)).$$

Direct computation shows

$$((\delta_0 \otimes f) * A)\varphi(x) = \int_G \hat{f}(y - x) k_A(x, y) \varphi(y) \, \mathrm{d}y$$

for $f \in L^1(\widehat{G})$, $\varphi \in L^2(G)$, $x \in G$.
$\implies \mathcal{C}_{0,1}(G) \subseteq \mathrm{BDO}(G)$.

Conversely, if $A \in \mathrm{BO}_K(G)$, choose $f \in L^1(\widehat{G})$ with $\hat{f} = 1$ on K. Then $A = (\delta_0 \otimes f) * A$, hence $A \in \mathcal{C}_{0,1}(G)$.

Theorem (Fulsche/H. '24)

Let G be an lca group. Then $\mathrm{BDO}(G) = \mathcal{C}_{0,1}(G)$.

Sketch of the proof:

We know that

$$\mathcal{C}_1(G) = L^1(G \times \widehat{G}) * \mathcal{L}(L^2(G)).$$

Direct computation shows

$$((\delta_0 \otimes f) * A)\varphi(x) = \int_G \hat{f}(y - x) k_A(x, y) \varphi(y) \, \mathrm{d}y$$

for $f \in L^1(\widehat{G})$, $\varphi \in L^2(G)$, $x \in G$.

$\Longrightarrow \mathcal{C}_{0,1}(G) \subseteq \mathrm{BDO}(G)$.

Conversely, if $A \in \mathrm{BO}_K(G)$, choose $f \in L^1(\widehat{G})$ with $\hat{f} = 1$ on K. Then $A = (\delta_0 \otimes f) * A$, hence $A \in \mathcal{C}_{0,1}(G)$.

Theorem (Fulsche/H. '24)

Let G be an lca group. Then $\mathrm{BDO}(G) = \mathcal{C}_{0,1}(G)$.

Sketch of the proof:

Similarly,

$$\mathcal{C}_{0,1}(G) = (\delta_0 \otimes L^1(\widehat{G})) * \mathcal{L}(L^2(G)).$$

Direct computation shows

$$((\delta_0 \otimes f) * A)\varphi(x) = \int_G \hat{f}(y-x)k_A(x,y)\varphi(y)\,\mathrm{d}y$$

for $f \in L^1(\widehat{G})$, $\varphi \in L^2(G)$, $x \in G$.

$\Longrightarrow \mathcal{C}_{0,1}(G) \subseteq \mathrm{BDO}(G)$.

Conversely, if $A \in \mathrm{BO}_K(G)$, choose $f \in L^1(\widehat{G})$ with $\hat{f} = 1$ on K. Then $A = (\delta_0 \otimes f) * A$, hence $A \in \mathcal{C}_{0,1}(G)$.

Theorem (Fulsche/H. '24)

Let G be an lca group. Then $\mathrm{BDO}(G) = \mathcal{C}_{0,1}(G)$.

Sketch of the proof:

Similarly,

$$\mathcal{C}_{0,1}(G) = (\delta_0 \otimes L^1(\widehat{G})) * \mathcal{L}(L^2(G)).$$

Direct computation shows

$$\big((\delta_0 \otimes f) * A\big)\varphi(x) = \int_G \hat{f}(y-x)k_A(x,y)\varphi(y)\,\mathrm{d}y$$

for $f \in L^1(\widehat{G})$, $\varphi \in L^2(G)$, $x \in G$.

$\Longrightarrow \mathcal{C}_{0,1}(G) \subseteq \mathrm{BDO}(G)$.

Conversely, if $A \in \mathrm{BO}_K(G)$, choose $f \in L^1(\widehat{G})$ with $\hat{f} = 1$ on K. Then $A = (\delta_0 \otimes f) * A$, hence $A \in \mathcal{C}_{0,1}(G)$.

Theorem (Fulsche/H. '24)

Let G be an lca group. Then $\mathrm{BDO}(G) = \mathcal{C}_{0,1}(G)$.

Sketch of the proof:

Similarly,

$$\mathcal{C}_{0,1}(G) = (\delta_0 \otimes L^1(\widehat{G})) * \mathcal{L}(L^2(G)).$$

Direct computation shows

$$\big((\delta_0 \otimes f) * A\big)\varphi(x) = \int_G \hat{f}(y - x) k_A(x, y)\varphi(y)\,\mathrm{d}y$$

for $f \in L^1(\widehat{G})$, $\varphi \in L^2(G)$, $x \in G$.

$\Longrightarrow \mathcal{C}_{0,1}(G) \subseteq \mathrm{BDO}(G)$.

Conversely, if $A \in \mathrm{BO}_K(G)$, choose $f \in L^1(\widehat{G})$ with $\hat{f} = 1$ on K. Then $A = (\delta_0 \otimes f) * A$, hence $A \in \mathcal{C}_{0,1}(G)$.

Theorem (Fulsche/H. '24)

Let G be an lca group. Then $\mathrm{BDO}(G) = \mathcal{C}_{0,1}(G)$.

Sketch of the proof:

Similarly,

$$\mathcal{C}_{0,1}(G) = (\delta_0 \otimes L^1(\widehat{G})) * \mathcal{L}(L^2(G)).$$

Direct computation shows

$$\big((\delta_0 \otimes f) * A\big)\varphi(x) = \int_G \hat{f}(y-x) k_A(x,y)\varphi(y)\,\mathrm{d}y$$

for $f \in L^1(\widehat{G})$, $\varphi \in L^2(G)$, $x \in G$.

$\Longrightarrow \mathcal{C}_{0,1}(G) \subseteq \mathrm{BDO}(G)$.

Conversely, if $A \in \mathrm{BO}_K(G)$, choose $f \in L^1(\widehat{G})$ with $\hat{f} = 1$ on K. Then $A = (\delta_0 \otimes f) * A$, hence $A \in \mathcal{C}_{0,1}(G)$.

For $A \in \mathcal{L}(L^2(G))$ and $H \subseteq G$ define

$$|||A|||_H := \sup \{ \|Af\| : \|f\| = 1, \operatorname{supp} f \subseteq gH \text{ for some } g \in G \}.$$

Theorem (Fulsche/H. '24)

For all $K \subseteq G$ compact and $c \in (0,1)$ there is a compact subset $H \subseteq G$ such that for all $T \in \mathrm{BO}_K(G)$ we have

$$|||T|||_H \geq c \|T\|.$$

lower norm: $\nu(A) := \inf_{\|\varphi\|=1} \|A\varphi\| = \left\|A^{-1}\right\|^{-1}$

Proposition

Let $A \in \mathcal{C}_1(G)$, $\Xi := G \times \widehat{G}$. Then there exists $y \in \partial\Xi$ such that

$$\nu(\alpha_y(A)) = \inf_{x \in \partial\Xi} \nu(\alpha_x(A)).$$

For $A \in \mathcal{L}(L^2(G))$ and $H \subseteq G$ define

$$|||A|||_H := \sup\{\|Af\| : \|f\| = 1, \operatorname{supp} f \subseteq gH \text{ for some } g \in G\}.$$

Theorem (Fulsche/H. '24)

For all $K \subseteq G$ compact and $c \in (0,1)$ there is a compact subset $H \subseteq G$ such that for all $T \in \mathrm{BO}_K(G)$ we have

$$|||T|||_H \geq c\,\|T\|.$$

lower norm: $\nu(A) := \inf_{\|\varphi\|=1} \|A\varphi\| = \left\|A^{-1}\right\|^{-1}$

Proposition

Let $A \in \mathcal{C}_1(G)$, $\Xi := G \times \widehat{G}$. Then there exists $y \in \partial\Xi$ such that

$$\nu(\alpha_y(A)) = \inf_{x \in \partial\Xi} \nu(\alpha_x(A)).$$

For $A \in \mathcal{L}(L^2(G))$ and $H \subseteq G$ define

$$\left\|\!\left|A\right|\!\right\|_H := \sup\left\{\|Af\| : \|f\| = 1, \operatorname{supp} f \subseteq gH \text{ for some } g \in G\right\}.$$

Theorem (Fulsche/H. '24)

For all $K \subseteq G$ compact and $c \in (0,1)$ there is a compact subset $H \subseteq G$ such that for all $T \in \mathrm{BO}_K(G)$ we have

$$\left\|\!\left|T\right|\!\right\|_H \geq c\,\|T\|.$$

lower norm: $\nu(A) := \inf\limits_{\|\varphi\|=1} \|A\varphi\| = \left\|A^{-1}\right\|^{-1}$

Proposition

Let $A \in \mathcal{C}_1(G)$, $\Xi := G \times \widehat{G}$. Then there exists $y \in \partial\Xi$ such that

$$\nu(\alpha_y(A)) = \inf_{x \in \partial\Xi} \nu(\alpha_x(A)).$$

For $A \in \mathcal{L}(L^2(G))$ and $H \subseteq G$ define

$$|||A|||_H := \sup \{\|Af\| : \|f\| = 1, \operatorname{supp} f \subseteq gH \text{ for some } g \in G\}.$$

Theorem (Fulsche/H. '24)

For all $K \subseteq G$ compact and $c \in (0,1)$ there is a compact subset $H \subseteq G$ such that for all $T \in \mathrm{BO}_K(G)$ we have

$$|||T|||_H \geq c\,\|T\|.$$

lower norm: $\nu(A) := \inf_{\|\varphi\|=1} \|A\varphi\| = \left\|A^{-1}\right\|^{-1}$

Proposition

Let $A \in \mathcal{C}_1(G)$, $\Xi := G \times \widehat{G}$. Then there exists $y \in \partial\Xi$ such that

$$\nu(\alpha_y(A)) = \inf_{x \in \partial\Xi} \nu(\alpha_x(A)).$$

Theorem (Fulsche/H. '24)

Let $A \in \mathcal{C}_1(G)$ *and* $\Xi := G \times \widehat{G}$. A *is Fredholm if and only all of its limit operators* $\alpha_x(A)$, $x \in \partial\Xi$, *are invertible. Moreover,*

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{x \in \partial\Xi} \mathrm{sp}(\alpha_x(A)).$$

Theorem (Fulsche/H. '24)

For $A \in \mathcal{C}_1(G)$ *we have*

$$\left\| A + \mathcal{K}(L^2(G)) \right\| = \max_{x \in \partial\Xi} \left\| \alpha_x(A) \right\|.$$

Theorem (Fulsche/H. '24)

Let $A \in \mathcal{C}_1(G)$ and $\Xi := G \times \widehat{G}$. A is Fredholm if and only all of its limit operators $\alpha_x(A)$, $x \in \partial\Xi$, are invertible. Moreover,

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{x \in \partial\Xi} \mathrm{sp}(\alpha_x(A)).$$

Theorem (Fulsche/H. '24)

For $A \in \mathcal{C}_1(G)$ we have

$$\left\| A + \mathcal{K}(L^2(G)) \right\| = \max_{x \in \partial\Xi} \left\| \alpha_x(A) \right\|.$$

Outlook:

- phase spaces Ξ that are not of the form $G \times \widehat{G}$, only assume that the representation

$$U_{\cdot} : \Xi \to \mathcal{L}(\mathcal{H}), \quad \xi \to U_\xi$$

  is integrable
- for fixed window $\varphi_0 \in \mathcal{H}$ with $\xi \mapsto \langle \varphi_0, U_\xi \varphi_0 \rangle \in L^1(\Xi)$ consider the wavelet transform

$$\mathcal{W}_{\varphi_0} : \mathcal{H} \to L^2(\Xi), \quad \mathcal{W}_{\varphi_0} f(\xi) := \langle f, U_\xi \varphi_0 \rangle$$

- consider operators on $\mathcal{W}_{\varphi_0}(\mathcal{H})$ ($\to$ Fock spaces)
- allows to consider operators on coorbit/modulation spaces ($p \neq 2$)

Outlook:

- phase spaces Ξ that are not of the form $G \times \widehat{G}$, only assume that the representation

$$U_\cdot : \Xi \to \mathcal{L}(\mathcal{H}), \quad \xi \to U_\xi$$

is integrable
- for fixed window $\varphi_0 \in \mathcal{H}$ with $\xi \mapsto \langle \varphi_0, U_\xi \varphi_0 \rangle \in L^1(\Xi)$ consider the wavelet transform

$$\mathcal{W}_{\varphi_0} : \mathcal{H} \to L^2(\Xi), \quad \mathcal{W}_{\varphi_0} f(\xi) := \langle f, U_\xi \varphi_0 \rangle$$

- consider operators on $\mathcal{W}_{\varphi_0}(\mathcal{H})$ ($\to$ Fock spaces)
- allows to consider operators on coorbit/modulation spaces ($p \neq 2$)

Outlook:

- phase spaces Ξ that are not of the form $G \times \widehat{G}$, only assume that the representation
$$U_\cdot : \Xi \to \mathcal{L}(\mathcal{H}), \quad \xi \to U_\xi$$
is integrable
- for fixed window $\varphi_0 \in \mathcal{H}$ with $\xi \mapsto \langle \varphi_0, U_\xi \varphi_0 \rangle \in L^1(\Xi)$ consider the wavelet transform
$$\mathcal{W}_{\varphi_0} : \mathcal{H} \to L^2(\Xi), \quad \mathcal{W}_{\varphi_0} f(\xi) := \langle f, U_\xi \varphi_0 \rangle$$
- consider operators on $\mathcal{W}_{\varphi_0}(\mathcal{H})$ ($\to$ Fock spaces)
- allows to consider operators on coorbit/modulation spaces ($p \neq 2$)

Outlook:

- phase spaces Ξ that are not of the form $G \times \widehat{G}$, only assume that the representation
$$U_\cdot : \Xi \to \mathcal{L}(\mathcal{H}), \quad \xi \to U_\xi$$
is integrable
- for fixed window $\varphi_0 \in \mathcal{H}$ with $\xi \mapsto \langle \varphi_0, U_\xi \varphi_0 \rangle \in L^1(\Xi)$ consider the wavelet transform
$$\mathcal{W}_{\varphi_0} : \mathcal{H} \to L^2(\Xi), \quad \mathcal{W}_{\varphi_0} f(\xi) := \langle f, U_\xi \varphi_0 \rangle$$
- consider operators on $\mathcal{W}_{\varphi_0}(\mathcal{H})$ ($\to$ Fock spaces)
- allows to consider operators on coorbit/modulation spaces ($p \neq 2$)

Define $V_\xi := \mathcal{W}_{\varphi_0} U_\xi \mathcal{W}_{\varphi_0}^*$,

$$\alpha_\xi(A) := V_\xi A V_\xi^*$$

for $A \in \mathcal{L}(\mathcal{W}_{\varphi_0}(\mathcal{H}))$, and

$$\mathcal{C}_1(\varphi_0) := \{A \in \mathcal{L}(\mathcal{W}_{\varphi_0}(\mathcal{H})) : \xi \mapsto \alpha_\xi(A) \text{ continuous}\} .$$

Theorem (Fulsche/H. '24)

Let Ξ be an lca group and denote the orthogonal projection of $L^2(\Xi)$ onto $\mathcal{W}_{\varphi_0}(\mathcal{H})$ by P_{φ_0}. Then

$$\mathcal{C}_1(\varphi_0) = P_{\varphi_0} \operatorname{BDO}(\Xi) P_{\varphi_0}.$$