Modulation spaces, harmonic analysis and pseudo-differential operators

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Plan of the talk

1. Schatten-von Neumann classes
2. Pseudo-differential operators (ΨDO)
3. Modulation spaces
4. Continuity of pseudo-differential operators
5. Feichtinger's minimization property in the quasi-Banach case
6. Compositions of pseudo-differential operators

The talk is based on joint works with D. Bhimani, Y. Chen, E. Cordero, A. Holst, and P. Wahlberg.

Some papers

The talk is based on the following papers.

- J. Toft *Continuity properties for non-commutative convolution algebras with applications in pseudo-differential calculus*, Bull. Sci. Math. **126** (2002), 115–142.
- A. Holst, J. Toft, P. Wahlberg *Weyl product algebras and modulation spaces*, J. Funct. Anal. **251** (2007), 463–491.
- E. Cordero, J. Toft, P. Wahlberg *Sharp results for the Weyl product on modulation spaces*, J. Funct. Anal. **267** (2014), 3016–3057.
- Y. Chen, J. Toft, P. Wahlberg *The Weyl product on quasi-Banach modulation spaces* Bull. Math. Sci. **9** (2019), 1950018–1.
- J. Toft *Schatten properties, nuclearity and minimality of phase shift invariant spaces*, Appl. Comput. Harmon. Anal. **46** (2019), 154–176.
- D. Bhimani, J. Toft *Factorizations for quasi-Banach time-frequency spaces and Schatten classes* (Preprint), arXiv:2307.01590.

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$$\mathscr{I}_{p_2}(B_2, B_3) \circ \mathscr{I}_{p_1}(B_1, B_2) \subseteq \mathscr{I}_{p_0}(B_1, B_3), \quad \frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}.$$

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- We put $\mathscr{I}_p = \mathscr{I}_p(L^2(\mathbf{R}^d), L^2(\mathbf{R}^d))$.

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Let $B_1 = B_2 = L^2$, $p = 1/100$ and $T \in \mathscr{I}_p$ be self-adjoint.

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Let $B_1 = B_2 = L^2$, $p = 1/100$ and $T \in \mathscr{I}_p$ be self-adjoint.

Then T is compact and for its eigenvalues: $|\lambda_j| \lesssim j^{-100}$

Nuclear operators

Let $\mathscr{B}_0$ be a Banach space with dual $\mathscr{B}_0'$, $\mathscr{B}$ be a quasi-Banach space, $r \in (0,1]$ and let $T \in \mathcal{B}(\mathscr{B}_0, \mathscr{B})$. Then T is called *r-nuclear*, if there are $\{\varepsilon_j\}_{j=1}^\infty \subseteq \mathscr{B}_0'$ and $\{e_j\}_{j=1}^\infty \subseteq \mathscr{B}$ such that

$$T = \sum_{j=1}^\infty e_j \otimes \varepsilon_j \quad \text{and} \quad \sum_{j=1}^\infty \|\varepsilon_j\|_{\mathscr{B}_0'}^r \|e_j\|_{\mathscr{B}}^r < \infty.$$

Let $\mathscr{N}_r(\mathscr{B}_0, \mathscr{B})$ be the set of r-nuclear operators, and

$$\|T\|_{\mathscr{N}_r(\mathscr{B}_0,\mathscr{B})} \equiv \inf \left(\sum_{j=1}^\infty \|\varepsilon_j\|_{\mathscr{B}_0'}^r \|e_j\|_{\mathscr{B}}^r \right)^{\frac{1}{r}}.$$

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If $\mathscr{B}_0$ and $\mathscr{B}$ are Hilbert spaces, then $\mathscr{N}_r(\mathscr{B}_0, \mathscr{B}) = \mathscr{I}_r(\mathscr{B}_0, \mathscr{B})$. Otherwise this equality might fail to hold.

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For example, $\mathscr{N}_r(\mathscr{B}_0, \mathscr{B})$ increases with r and $\mathscr{B}$ and decreases with $\mathscr{B}_0$.

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If $\quad T \in \mathscr{N}_r(\mathscr{B}_0, \mathscr{B}), \quad T_1 \in \mathcal{B}(\mathscr{B}_1, \mathscr{B}_0) \quad$ and $\quad T_2 \in \mathcal{B}(\mathscr{B}, \mathscr{B}_2)$,

then $\qquad T_2 \circ T \circ T_1 \in \mathscr{N}_r(\mathscr{B}_1, \mathscr{B}_2)$.

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If $\mathscr{H}_j$ are Hilbert spaces, then

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Weyl quantization: when $A = \frac{1}{2}I$, i. e. $\operatorname{Op}^w(a) = \operatorname{Op}_{\frac{1}{2}I}(a)$.

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If $A = \frac{1}{2} \cdot I_d$, i.e. the Weyl case, then we write $s_{A,p} = s_p^w$.

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The definition of $\operatorname{Op}_A(a)$ extends to any $a\in \mathscr{S}'(\mathbf{R}^{2d})$, and then

$$\operatorname{Op}_A(a) : \mathscr{S}(\mathbf{R}^d)\to \mathscr{S}'(\mathbf{R}^d) \quad \text{(continuous)}.$$

Kernel property: By Fourier inversion formula and Schwartz kernel theorem, it follows that for any linear and continuous map T from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ there is a unique $a\in \mathscr{S}'(\mathbf{R}^{2d})$ s.t. $T=\operatorname{Op}_A(a)$.

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Definition of ΨDO

Let A be a fixed real $d \times d$ matrix, $a \in \mathscr{S}'(\mathbf{R}^{2d})$. Then

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By kernel property it follows that $a \mapsto \operatorname{Op}_A(a)$ is an isometric bijection.

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Consequently, Schatten issues can be transfered from operators to symbols.

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$$\mathscr{F}_\sigma a(X) = \pi^{-d} \int_{\mathbf{R}^{2d}} a(Y) e^{2i\sigma(X,Y)}\, dY.$$

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- Rank-one Weyl operators and Wigner distributions:

$$\mathrm{Op}^w(a)f(x) = \big((2\pi)^{-\frac{d}{2}}(f,g_2)_{L^2}\big)g_1(x),$$

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- Spectral theorem for Weyl calculus: Suppose $p<\infty$. Then $a \in s_p^w(\mathbf{R}^{2d})$, iff

$$a = \sum_{j=1}^{\infty}\lambda_j W_{f_j,g_j}, \quad \{f_j\},\{g_j\}\in \mathrm{ON},\ \lambda_j \ge \lambda_{j+1}\ge 0,\ \{\lambda_j\}\in \ell^p(\mathbf{Z}_+).$$

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and $a\#b$ is defined by $\operatorname{Op}^w(a\#b) = \operatorname{Op}^w(a)\circ\operatorname{Op}^w(b)$, then

$$a\#b = (2\pi)^{-\frac{d}{2}} a *_\sigma (\mathscr{F}_\sigma b), \qquad \mathscr{F}_\sigma(a *_\sigma b) = (\mathscr{F}_\sigma a) *_\sigma b = \check{a} *_\sigma (\mathscr{F}_\sigma b).$$

Convolution properties Schatten classes

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The convolution results by Werner from 1984 can now be formulated as follows.

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Here

$$(a*b)(X) = \int_{\mathbf{R}^{2d}} a(X-Y)b(Y)\,dY, \quad X = (x,\xi) \in \mathbf{R}^{2d}$$

the usual convolution.

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The convolution results by Werner from 1984 can now be formulated as follows.

Prop. (Werner 1984 - present formulation T. 1996)

Suppose $p_j \in [1,\infty]$, $j = 0,1,2$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p_0}$. Then

$$s_{p_1}^w * L^{p_2} \subseteq s_{p_0}^w \quad \text{and} \quad s_{p_1}^w * s_{p_2}^w \subseteq L^{p_0}.$$

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Extensions to other pseudo-differential calculi.

Prop.

Suppose $p_j \in [1,\infty]$, $j = 0,1,2$, and $A, B \in \mathbf{R}^{d\times d}$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p_0}$ and $A + B = I_d$. Then

$$s_{A,p_1} * L^{p_2} \subseteq s_{A,p_0} \quad \text{and} \quad s_{A,p_1} * s_{B,p_2} \subseteq L^{p_0}.$$

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Extensions to exponents smaller than 1. Here

$$WL^{p,q}(\mathbf{R}^d) = \{\, f\, ;\, \|f\|_{WL^{p,q}} < \infty \,\},$$

$$\|f\|_{WL^{p,q}} = \Big\| \{ \|f\|_{L^p(j+Q)} \}_{j \in \mathbf{Z}^d} \Big\|_{\ell^q(\mathbf{Z}^d)}, \quad Q = [0,1]^d.$$

Prop. (e.g. Bhimani, T. 2023)

Suppose $p \in (0,1]$. Then $s_{A,p} * WL^{1,p} \subseteq s_{A,p}$.

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Another convolution/multiplication result.

Prop. (T. 1996)

Suppose $p_j \in [1,\infty]$, $t_j \in \mathbf{R} \setminus 0$, $j = 0,\dots,N$ satisfy $\frac{1}{p_1} + \cdots + \frac{1}{p_N} = N - 1 + \frac{1}{p_0}$, and let $a_j \in s^w_{p_j}$. Then the following is true:

1. if $\pm t_1^2 \pm \cdots \pm t_N^2 = 1$, then $a_0 = a_1(t_1\,\cdot\,)\cdots a_N(t_N\,\cdot\,) \in s^w_{p_0}$;
2. if $\pm t_1^{-2} \pm \cdots \pm t_N^{-2} = 1$, then $a_0 = a_1(t_1\,\cdot\,) * \cdots * a_N(t_N\,\cdot\,) \in s^w_{p_0}$. Moreover, if $\operatorname{Op}^w(a_j) \geq 0$ (as operator) for every $j = 1,\dots,N$, then $\operatorname{Op}^w(a_j) \geq 0$.

Some consequences

$$s_p^w * L^q \subseteq s_r^w, \quad s_p^w * s_q^w \subseteq L^r,$$

$$a(s\,\cdot\,) * b(t\,\cdot\,) \in s_r^w, \quad a \in s_p^w,\ b \in s_q^w, \quad \tfrac{1}{p} + \tfrac{1}{q} = 1 + \tfrac{1}{r}, \qquad \text{(*)}$$

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- Toeplitz operators: Let $\phi \in L^2(\mathbf{R}^d)$ be fixed. Then define $\operatorname{Tp}_\phi(a)$ by

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    Moreover, $\operatorname{Tp}_\phi(a(\sqrt{2}\,\cdot\,)) \ge 0$ when $\operatorname{Op}^w(a) \ge 0$ as operators.

Modulation spaces - Preparations

- Window function: $\phi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$.
- Weight function: $0 < \omega \in L^\infty_{loc}(\mathbf{R}^{2d})$.

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- Fourier Transform: (FT)

$$\widehat{f}(\xi) = \mathscr{F} f(\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y) e^{-i\langle y,\xi \rangle}\, dy.$$

- Short-Time Fourier Transform: (STFT)

$$\begin{aligned} V_\phi f(x,\xi) &= \mathscr{F}(f\, \overline{\phi(\,\cdot\, - x)})(\xi) \\ &= (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)} e^{-i\langle y,\xi \rangle}\, dy. \end{aligned}$$

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Let $p, q \in (0, \infty]$. Recall that

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- f in $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, iff

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \Big(\int \Big(\int |V_\phi f(x,\xi)\omega(x,\xi)|^p \, dx \Big)^{q/p} \, d\xi \Big)^{1/q} < \infty$$

Weight functions

Let $\omega, v > 0$ on $\mathbf{R}^d$ (or on $\mathbf{R}^{2d}$).

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- $\mathscr{P}(\mathbf{R}^d)$ is the set of all functions on $\mathbf{R}^d$ moderated by some polynomials.

  Staying in the usual distribution theory $\iff$ $\omega \in \mathscr{P}(\mathbf{R}^d)$.

ΨDO with symbols in modulation spaces

An "average" result here is:

Thm 1 - T, 2017

Let $p, q, p_j, q_j \in (0, \infty]$, $a \in M^{p,q}(\mathbf{R}^{2d})$, $q \le p_2, q_2 \le p$, *and*

$$\frac{1}{p_1} - \frac{1}{p_2} = \frac{1}{q_1} - \frac{1}{q_2} = 1 - \frac{1}{p} - \frac{1}{\max(1,q)}.$$

Then $\operatorname{Op}_A(a) : M^{p_1,q_1}(\mathbf{R}^d) \to M^{p_2,q_2}(\mathbf{R}^d)$.

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The case $p, q, p_j, q_j \ge 1$ was obtained already during 2003–2004 by Gröchenig-Heil and T.

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The result holds for weighted modulation spaces as well.

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For suitable choices of p_j, q_j we obtain

$$\operatorname{Op}_A(a) : M^{p,q} \to M^{p,q}$$

continuous, when $a \in M^{\infty,q_0}$, $q_0 \in (0,1]$, $p, q \in [q_0, \infty]$.

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Thm 1 - T, 2017

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This gives Calderon-Valiancourt's theorem:

$$\operatorname{Op}_A(a) : L^2 \to L^2, \quad a \in S^0_{0,0} \subseteq M^{\infty,q_0}.$$

Note that: $M^{2,2} = L^2$ and $S^0_{0,0}(\mathbf{R}^{2d}) = \{\, a \in C^\infty(\mathbf{R}^{2d})\,;\, \partial^\alpha a \in L^\infty \,\}$.

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Improvement concerning conditions on $p, q, p_j, q_j \in [1, \infty]$ by Cordero-Nicola (2018).

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General result containing all these cases as well as suitable Orlicz modulation spaces are obtained by Gumber, Rana, T., Üster (2024).

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We have/had:

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Interpolations of these results give:

Thm Sjöstrand 1994, Gröchenig-Heil 1997, T. 2003

Let $p, q_1, q_2 \in (0, \infty]$ be such that $q_1 \le \min(p, p')$ and $q_2 \le \max(p, p')$. Then

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Natural extensions to weighted spaces exist (e.g. by T. 2007–).

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Thm (T. 2018)

Let $p \in (0,1]$, $\mathscr{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$ be a quasi-Banach space such that:

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When proving the theorem one consider non-uniform Gabor expansions

Minimality of matrix classes

Let $p \in (0,1]$, J be an index set and $\mathbb{U}^p(J)$ be the set of all $J \times J$-matrices $A = (a(j,k))_{j,k\in J}$ such that

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2. $A_{j_0,k_0} \equiv (\delta_{j,j_0}\delta_{k,k_0})_{j,k\in J} \in \mathscr{B}, \qquad \sup_{j_0,k_0\in J} \|A_{j_0,k_0}\|_{\mathscr{B}} < \infty.$

Then $\mathbb{U}^p(J) \subseteq \mathscr{B}$.

Proof:

Minimality of matrix classes

Let $p \in (0,1]$, J be an index set and $\mathbb{U}^p(J)$ be the set of all $J \times J$-matrices $A = (a(j,k))_{j,k\in J}$ such that

$$\|A\|_{\mathbb{U}^p}^p \equiv \sum_{j,k\in J} |a(j,k)|^p < \infty.$$

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Proof: By formal computations,

$$\|A\|_{\mathscr{B}}^p = \Big\|\sum_{j,k} a(j,k)A_{j,k}\Big\|_{\mathscr{B}}^p \le \sum_{j,k} |a(j,k)|^p \|A_{j,k}\|_{\mathscr{B}}^p$$

$$\le C^p \sum_{j,k} |a(j,k)|^p = C^p \|A\|_{\mathbb{U}^p}^p,$$

Applications - Schatten and nuclear results on modulation spaces

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Hence by Thm: $\quad M^p \subseteq s_{A,p}, \quad 0 < p \leq 1.$

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The minimality of $\mathbb{U}^p(J)$ gives $\mathbb{U}^p(J) \subseteq \mathscr{N}_p(\ell^\infty(J), \ell^p(J))$. The theorem now follows by using Gabor analysis, which carry over the discrete results to modulation space results.

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J. Delgado, M. Ruzhansky and collaborators have obtained several related results.

Compositions of Ψdo

Recall:

- $S^0_{0,0}(\mathbf{R}^{2d})$ is the set of all $a \in C^\infty(\mathbf{R}^{2d})$ such that

$$\partial^\alpha a \in L^\infty(\mathbf{R}^{2d}), \qquad \forall \alpha.$$

- $a\#b$ is defined by

$$\operatorname{Op}^w(a\#b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b)$$

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- Labate 2001: If $p \in [1,2]$, then

$$M^{p,p}_{(\omega_r)}\#M^{p,p}_{(\omega_r)} = M^{p,p}_{(\omega_r)}, \quad \omega_r(x,\xi) = (1+|x|+|\xi|)^r,\ r \ge 0.$$

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- Gröchenig 2006: $M^{\infty,1}_{(v)}\#M^{\infty,1}_{(v)} = M^{\infty,1}_{(v)}$, v submultiplicative.

Compositions of Ψdo - extensions

$$\mathrm{Op}^w(a\#b) = \mathrm{Op}^w(a) \circ \mathrm{Op}^w(b), \qquad \mathsf{R}(u) = \sum_{j=0}^{2} u_j - 1, \quad u = (u_0, u_1, u_2) \in [0,1]^3$$

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Thm 1 (Holst, T., Wahlberg 2007)

Let $\omega_j \in \mathscr{P}$ "be suitable," and assume

$$\mathsf{R}(\tfrac{1}{p}) = \mathsf{R}(\tfrac{1}{q'}), \qquad q_1,q_2 \le q_0', \quad 0 \le \mathsf{R}(\tfrac{1}{q'}) \le \tfrac{1}{p_j}, \tfrac{1}{q_j}, \tfrac{1}{p_0'}, \tfrac{1}{q_0'} \le 1-\mathsf{R}(\tfrac{1}{q'}), \quad j=1,2.$$

Then $\quad M^{p_1,q_1}_{(\omega_1)} \# M^{p_2,q_2}_{(\omega_2)} \subseteq M^{p_0',q_0'}_{(1/\omega_0)}$.

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Thm 1 essentially contains all previous composition results.

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Thm 2 (Cordero, T., Wahlberg 2014)

Let $\omega_j \in \mathscr{P}_E$ "be suitable," and assume

$$\max\left(\mathsf{R}(\tfrac{1}{q'}),0\right) \le \min_{j=0,1,2}\left(\tfrac{1}{p_j},\tfrac{1}{q_j'},\mathsf{R}(\tfrac{1}{p})\right).$$

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The proofs of Thm 1 and Thm 2 are based on the formula

$$V_{\phi\#\psi}(a\#b)(X,Y) = \pi^{-d} \int_{\mathbf{R}^{2d}} e^{2i\sigma(Z,Y)} V_\phi a(X-Y+Z, Z)\, V_\psi b(X+Z, Y-Z)\, dZ,$$

which follows by some Fourier analysis.

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Here symplectic STFT are used (also in definition of modulation spaces).

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Thm 2 is essentially sharp, and contains Thm 1 and all other results presented so far.

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Comparisons between the results

$M^{p_1,q_1} \# M^{p_2,q_2}$	(Thm 2) $M^{p_0',q_0'}$	(Thm 1) $M^{p_0',q_0'}$	$M^{p_1,q_1} \# M^{p_2,q_2}$	(Thm 2) $M^{p_0',q_0'}$	(Thm 1) $M^{p_0',q_0'}$
$M^{1,1} \# M^{1,1}$	$M^{1,1}$	—	$M^{1,1} \# M^{1,2}$	$M^{1,2}$	—
$M^{1,1} \# M^{1,\infty}$	$M^{1,\infty}$	—	$M^{1,1} \# M^{2,1}$	$M^{1,1}$	—
$M^{1,1} \# M^{2,2}$	$M^{1,2}$	—	$M^{1,1} \# M^{2,\infty}$	$M^{1,\infty}$	—
$M^{p,q} \# M^{\infty,1}$	$M^{p,q}$	$M^{p,q}$	$M^{1,1} \# M^{\infty,2}$	$M^{1,2}$	$M^{1,2}$
$M^{1,1} \# M^{\infty,\infty}$	$M^{1,\infty}$	$M^{1,\infty}$	$M^{2,2} \# M^{2,2}$	$M^{2,2}$, $M^{1,\infty}$	$M^{2,2}$, $M^{1,\infty}$
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$M^{1,2} \# M^{2,2}$	$M^{2,2}$, $M^{1,\infty}$	—	$M^{1,2} \# M^{\infty,2}$	$M^{1,\infty}$	$M^{1,\infty}$
$M^{2,1} \# M^{1,\infty}$	$M^{1,\infty}$	—	$M^{2,1} \# M^{2,1}$	$M^{1,1}$	$M^{1,1}$
$M^{2,1} \# M^{2,2}$	$M^{1,2}$	$M^{1,2}$	$M^{2,1} \# M^{2,\infty}$	$M^{1,\infty}$	$M^{1,\infty}$
$M^{2,1} \# M^{\infty,2}$	$M^{2,2}$	$M^{2,2}$	$M^{2,1} \# M^{\infty,\infty}$	$M^{2,\infty}$	$M^{2,\infty}$
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Thank you for your attention!