From Born, Jordan, and Heisenberg to Weyl, and back

Maurice de Gosson

Universität Wien
Institut für Mathematik, NUHAG

August 2024

The Problem of Quantization

In the early years of quantum mechanics physicists were confronted with an ordering problem: assume that some quantization process associates to the real variables x (position) and p (momentum) two operators $\widehat{x}$ and $\widehat{p}$ satisfying Max Born's canonical commutation rule $\widehat{x}\widehat{p} - \widehat{p}\widehat{x} = i\hbar$ $(\widehat{x}\psi = x\psi, \widehat{p}\psi = -i\hbar\partial_x)$.

The basic question which has led to the quantization problem in physics on one hand, and to the general theory of pseudo-differential calculus on the other hand is: *What should the operator $A = \mathrm{Op}(a)$ associated to a symbol $a(x,p)$ be? For instance, how do we quantize*

$$H(z,t) = \sum_{j=1}^{n} \frac{1}{2m_j} \left(p_j - A_j(x,t)\right)^2 + V(x,t) \tag{1}$$

where V is a scalar potential function, and $A = (A_1, ..., A_n)$ a vector potential?

Some usual quantization rules

- **Continuity**: Op is a continuous linear mapping

$$\mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n));$$

- **Triviality**: For $b = b(x)$ and $c = c(p)$

$$\mathrm{Op}(b \otimes 1)\psi(x) = b(x)\psi(x) \quad , \quad \mathrm{Op}(1 \otimes c)\psi(x) = F^{-1}(cF)\psi(x)$$

  hence $\widehat{x}\psi(x) = x\psi(x)$ and $\widehat{p}\psi(x) = -i\hbar\partial_x\psi(x)$;

- **Self-adjointness**:

$$\mathrm{Op}(a) \text{ self-adjoint} \Longleftrightarrow a \text{ is a real symbol;}$$

- "**Dirac's dream**":

$$[\mathrm{Op}(a), \mathrm{Op}(b)] = i\hbar\mathrm{Op}(\{a, b\})$$

- **Normal and anti-normal ordering**:

$$\mathrm{Op}_{\mathrm{N}}(x^m p^\ell) = \widehat{x}^m \widehat{p}^\ell \quad , \quad \mathrm{Op}_{\mathrm{AN}}(x^m p^\ell) = \widehat{p}^\ell \widehat{x}^m \tag{2}$$

- **Weyl rule** (1927); it is the most symmetrical:

$$\mathrm{Op}_{\mathrm{Weyl}}(x^m p^\ell) = \frac{1}{2^\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} \widehat{p}^{\ell-k} \widehat{x}^m \widehat{p}^k \tag{3}$$

- **Born and Jordan rule** (1925); it is the equally averaged ordering (McCoy):

$$\mathrm{Op}_{\mathrm{BJ}}(x^m p^\ell) = \frac{1}{\ell+1} \sum_{k=0}^{\ell} \widehat{p}^{\ell-k} \widehat{x}^m \widehat{p}^k . \tag{4}$$

BJ versus Weyl

Born–Jordan and Weyl rules coincide when $m+\ell \leq 2$: in both cases $xp \longrightarrow \frac{1}{2}(\widehat{x}\widehat{p}+\widehat{p}\widehat{x})$. But they are different as soon as $m \geq 2$ and $\ell \geq 2$. For instance

$$\mathrm{Op}_{\mathrm{W}}(x^2p^2) = \widehat{x}^2\widehat{p}^2 - 2i\hbar\widehat{x}\widehat{p} - \tfrac{1}{2}\hbar^2 \quad (5)$$

$$\mathrm{Op}_{\mathrm{BJ}}(x^2p^2) = \widehat{x}^2\widehat{p}^2 - 2i\hbar\widehat{x}\widehat{p} - \tfrac{2}{3}\hbar^2. \quad (6)$$

Thus: $\mathrm{Op}_{\mathrm{BJ}}(x^2p^2) \neq \mathrm{Op}_{\mathrm{W}}(x^2p^2)$.

The Angular Momentum Dilemma

Consider the square ℓ^2 of the classical angular momentum $\ell = \mathbf{r} \times \mathbf{p}$:

$$\ell^2 = (x_2 p_3 - x_3 p_2)^2 + (x_3 p_1 - x_1 p_3)^2 + (x_1 p_2 - x_2 p_1)^2.$$

We have

$$\mathrm{Op}_{\mathrm{BJ}}(\ell^2) - \mathrm{Op}_{\mathrm{W}}(\ell^2) = \tfrac{1}{2}\hbar^2,$$

This discrepancy explains the “angular momentum dilemma” (https://en.wikipedia.org/wiki/Geometric_quantization).
It is one indication, among others, that Born–Jordan quantization might indeed be the correct one in physics!

Why should BJ quantization be good for Physics?

The Born and Jordan 1925 paper *Zur Quantenmechanik, Z. Physik 34 (1925)* which rigorously justifies Heisenberg's "matrix mechanics" explicitly requires the use of the quantization rule

$$x^m p^\ell \longrightarrow \frac{1}{\ell+1}\sum_{k=0}^{\ell} \widehat{p}^{\ell-k}\widehat{x}^m\widehat{p}^k$$

(otherwise their constructions break down). Consequence: if one wants a unique physical quantum mechanics, that is, if one wants the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}\psi$$

to describe Nature in the same way as the Heisenberg picture does, then $\widehat{H}$ *must* be quantized following Born and Jordan's prescription (M. de Gosson, *Found. Phys. 44(10) (2014)*).

Physical Hamiltonians

In physics one uses the "Hamiltonian function"

$$H(z,t)=\sum_{j=1}^{n}\frac{1}{2m_j}\left(p_j-A_j(x,t)\right)^2+V(x,t) \tag{7}$$

where V is a scalar potential function, and $A=(A_1,...,A_n)$ a vector potential. One can show that

$$\mathrm{Op}_{\mathrm{BJ}}(H)=\mathrm{Op}_{\mathrm{W}}(H)=\widehat{H}$$

where $\widehat{H}$ is the usual quantization of H viewed as a symbol:

$$\widehat{H}=\sum_{j=1}^{n}\frac{1}{2m_j}\left(-i\hbar\partial_{x_j}-A_j(x,t)\right)^2+V(x,t).$$

An essential remark: consider Shubin's τ-ordering

$$\mathrm{Op}_\tau(x^m p^\ell) = \sum_{k=0}^{n} \binom{\ell}{k} (1-\tau)^k \tau^{\ell-k} \widehat{p}^{\ell-k} \widehat{x}^m \widehat{p}^k . \tag{8}$$

It coincides with the Weyl ordering when $\tau = \frac{1}{2}$, with the normal ordering when $\tau = 1$ and with the antinormal ordering when $\tau = 0$. We have, by the properties of the beta function,

$$\int_0^1 (1-\tau)^k \tau^{\ell-k} d\tau = \frac{(\ell-k)!k!}{(\ell+1)!}$$

hence the Born–Jordan ordering is the average over $[0,1]$ of the τ-orderings:

$$\mathrm{Op}_{\mathrm{BJ}}(x^m p^\ell) = \int_0^1 \mathrm{Op}_\tau(x^m p^\ell) d\tau.$$

This relation is the starting point of the theory of Born–Jordan pseudodifferential operators.

Shubin's Operators

Shubin's τ-pseudo-differential operator $A_\tau = \operatorname{Op}_\tau(a)$ is the continuous operator $A_\tau : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ with distributional kernel

$$K_{A_\tau}(x,y) = \left(\tfrac{1}{2\pi\hbar}\right)^{n/2} (F_2^{-1}a)((1-\tau)x + \tau y, x-y) \tag{9}$$

where F_2^{-1} is the inverse Fourier transform in the p variables. Formally:

$$A_\tau\psi(x) = \left(\tfrac{1}{2\pi\hbar}\right)^{n} \iint e^{\frac{i}{\hbar}p(x-y)} a((1-\tau)x + \tau y, p)\psi(y)dpdy. \tag{10}$$

- For $\tau = \frac{1}{2}$ we recover the Weyl correspondence:
- For $\tau = 0$ we get the standard pseudodifferential operator (normal ordering).

BJ Quantization: formal definition

The Born–Jordan operator $A_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a)$ (a in some symbol class) is (by definition) the average

$$A_{\mathrm{BJ}} = \int_0^1 A_\tau d\tau, \tag{11}$$

that is, formally,

$$A_{\mathrm{BJ}}\psi(x) = \left(\tfrac{1}{2\pi\hbar}\right)^n \iint e^{\frac{i}{\hbar}p(x-y)} b(x,y,p)\psi(y)dpdy$$

with

$$b(x,y) = \int_0^1 a((1-\tau)x + \tau y, p)d\tau.$$

All this needs to be expressed more rigorously... This will be done using the theory of the "Cohen class". Let jus first recall the definition of the Wigner transfor.

The Wigner transform

The cross-Wigner transform of $\psi, \phi \in L^2(\mathbb{R}^n)$ is defined by

$$W(\psi,\phi)(z) = \left(\tfrac{1}{\pi\hbar}\right)^n (\widehat{\Pi}(z)\psi|\phi)$$

where

$$\widehat{\Pi}(z) = \widehat{T}(z)\widehat{\Pi}\widehat{T}(z)^{-1} \ , \ \widehat{\Pi}\psi(x) = \psi(-x).$$

Explicitly, setting $W\psi = W(\psi,\psi)$:

$$\begin{aligned} W(\psi,\phi)(z) &= \left(\tfrac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py}\psi(x+\tfrac{1}{2}y)\overline{\phi(x-\tfrac{1}{2}y)}d^n y \\ W\psi(z) &= \left(\tfrac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py}\psi(x+\tfrac{1}{2}y)\overline{\psi(x-\tfrac{1}{2}y)}d^n y. \end{aligned}$$

The Cohen Class

Recall that the cross-Wigner transform of $(\psi, \phi) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ is

$$W(\psi,\phi)(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py}\psi(x+\tfrac{1}{2}y)\overline{\phi(x-\tfrac{1}{2}y)}dy. \qquad (12)$$

Also, the Heisenberg displacement operator is:

$$\widehat{T}(z_0)\psi(x) = e^{-\frac{i}{\hbar}\sigma(\hat{z},z_0)}\psi(x) = e^{\frac{i}{\hbar}(p_0x-\frac{1}{2}p_0x_0)}\psi(x-x_0).$$

Theorem (Gröchenig, 2001)

Let $Q : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$ *be a sesquilinear form. If*

$$Q(\psi,\phi)(z-z_0) = Q(\widehat{T}(z_0)\psi, \widehat{T}(z_0)\phi)(z) \qquad (13)$$

$$|Q(\psi,\phi)(0,0)| \leq ||\psi||\,||\phi|| \qquad (14)$$

for all ψ, ϕ *in* $L^2(\mathbb{R}^n)$ *then there exists* $\theta \in \mathcal{S}'(\mathbb{R}^{2n})$ *such that*

$$Q\psi = Q(\psi,\psi) = W\psi * \theta. \qquad (15)$$

Definition

Let $Q : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ be a sesquilinear form. We say that Q belongs to the *Cohen class* if we have

$$Q(\psi, \phi) = W(\psi, \phi) * \theta \tag{16}$$

for some distribution $\theta \in \mathcal{S}'(\mathbb{R}^{2n})$. The function θ is called a Cohen kernel.

The cross-Wigner transform trivially belongs to the Cohen class (take $\theta = \delta$, the Dirac distribution on $\mathbb{R}^{2n}$).
Another example is the Husimi transform: take $\theta(z) = (\pi\hbar)^{-n} e^{-\frac{1}{\hbar}|z|^2}$ (it is the Wigner transform of $\phi_0(x) = (\pi\hbar)^{-n/4} e^{-|x|^2/2\hbar}$). To every element Q of the Cohen class we can associate a pseudo-differential calculus: for a symbol $a \in \mathcal{S}(\mathbb{R}^{2n})$ we associate a operator $A_Q : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$:

$$\langle \widehat{A}_Q \psi, \overline{\phi} \rangle = \langle\langle a, Q(\psi, \phi) \rangle\rangle \tag{17}$$

for all $(\psi, \phi) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$. For instance, for $Q = W$ we recover the usual Weyl operators.

In particular, the Shubin pseudo-differential calculus is obtained by choosing as Cohen kernel

$$\theta_{(\tau)}(z) = \frac{2^n}{|2\tau - 1|^n} e^{\frac{2i}{\hbar(2\tau-1)}px} \ , \ \tau \neq \tfrac{1}{2}. \tag{18}$$

In fact, one shows that $W_\tau(\psi, \phi) = W(\psi, \phi) * \theta_{(\tau)}$ is given by

$$W_\tau(\psi, \phi)(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py} \psi(x + \tau y) \overline{\phi(x - (1-\tau)y)} dy$$

from which the usual formula

$$A_\tau \psi(x) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int e^{\frac{i}{\hbar}p(x-y)} a((1-\tau)x + \tau y, p) \psi(y) dp dy$$

readily follows after some calculations.

The Born–Jordan operators correspond to the choice of Cohen kernel

$$\theta_{\mathrm{BJ}} = \int_0^1 \theta_{(\tau)}(z)d\tau;$$

this kernel is "explicitly" given by

$$\theta_{\mathrm{BJ}} = \left(\tfrac{1}{2\pi\hbar}\right)^n F_\sigma \chi_{\mathrm{BJ}} \Longleftrightarrow F_\sigma \theta_{\mathrm{BJ}} = \left(\tfrac{1}{2\pi\hbar}\right)^n \chi_{\mathrm{BJ}}$$

where

$$\chi_{\mathrm{BJ}}(x,p) = \mathrm{sinc}\,(px/2\hbar) = \frac{\sin(px/2\hbar)}{px/2\hbar} \tag{19}$$

and $F_\sigma \chi_{\mathrm{BJ}}$ is its symplectic Fourier transform:

$$F_\sigma \chi_{\mathrm{BJ}}(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}\sigma(z,z')}\chi_{\mathrm{BJ}}(z')dz' = F\chi_{\mathrm{BJ}}(-Jz).$$

Harmonic analysis of BJ operators

A very useful way of writing a Weyl operator $A_{\mathrm{Weyl}} = Op_{\mathrm{W}}(a)$ (Shubin operator with $\tau = \frac{1}{2}$) is the "harmonic decomposition

$$A_{\mathrm{Weyl}} = \left(\tfrac{1}{2\pi\hbar}\right)^n \int a_\sigma(z)\widehat{T}(z)dz \tag{20}$$

where $\widehat{T}(z) = e^{-\frac{i}{\hbar}\sigma(\hat{z},z_0)}$ is the Heisenberg displacement operator and a_σ the symplectic Fourier transform of the symbol a. More generally the operator A_Q associated with an element of the Cohen class with kernel Θ can be written

$$A_Q = \left(\tfrac{1}{2\pi\hbar}\right)^n \int a_\sigma(z)\widehat{T}_Q(z)d^{2n}z.$$

where

$$\widehat{T}_Q(z) = \widehat{T}(z)(\Theta^\vee)_\sigma(z) \ \ , \ \ \Theta^\vee(z) = \Theta(-z)).$$

In the BJ case this leads to:

Theorem

(i) Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. The Born–Jordan operator $A_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a)$ is given by

$$A_{\mathrm{BJ}}\psi = \left(\tfrac{1}{2\pi\hbar}\right)^n \int a_\sigma(z)\chi_{\mathrm{BJ}}(z)\widehat{T}(z)\psi d^{2n}z \tag{21}$$

where χ_{BJ} is defined as above by

$$\chi_{\mathrm{BJ}}(z) = \mathrm{sinc}(px/2\hbar). \tag{22}$$

(ii) In particular the operator A_{BJ} is a continuous operator $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ for every $a \in \mathcal{S}'(\mathbb{R}^{2n})$.

It follows, by Schwartz's kernel theorem, that A_{BJ} is a Weyl operator: we have $\mathrm{Op}_{\mathrm{BJ}}(a) = \mathrm{Op}_{\mathrm{W}}(b)$ for some $b \in \mathcal{S}'(\mathbb{R}^{2n})$ satisfying $b_\sigma(z) = a_\sigma\chi_{\mathrm{BJ}}(z)$: **Division problem**! We will come back to this in a moment.

The question of symplectic covariance

A classical property of Weyl operators is their covariance with respect to symplectic linear transformations. In fact if $A = Op_{\mathrm{W}}(a)$ and $s \in \mathrm{Sp}(n)$ then

$$SOp_{\mathrm{W}}(a)S^{-1} = Op_{\mathrm{W}}(a \circ s^{-1}) \tag{23}$$

where $S \in \mathrm{Mp}(n)$ is anyone of the two metaplectic operators $\pm S$ covering s. Recall that the symplectic group $\mathrm{Sp}(n)$ consists of all $s \in GL(2n, \mathbb{R})$ such that $s^*\sigma = \sigma$ and $\mathrm{Mp}(n)$ is the unitary representation of the two-fold covering group $\mathrm{Sp}_2(n)$ of $\mathrm{Sp}(n)$. It turns out that Weyl calculus is the only pseudo-differential calculus satisfying this property, hence BJ calculus will not be fully symplectic covariant. However:

$$SOp_{\mathrm{BJ}}(a)S^{-1} = Op_{\mathrm{BJ}}(a \circ s^{-1}) \tag{24}$$

for all $s \in \mathrm{Sp}_0(n)$ (the subgroup of $\mathrm{Sp}(n)$ generated by $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and the $\begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}$.

BJ quantization and Shubin classes

The Shubin symbol casses are defined as follows:

Definition

Let $m \in \mathbb{R}$ and $0 < \rho \leq 1$. The symbol class $\Gamma_\rho^m(\mathbb{R}^{2n})$ consists of all $a \in C^\infty(\mathbb{R}^{2n})$ such tat for every $\alpha \in \mathbb{N}^{2n}$ there exists $C_\alpha \geq 0$ with

$$|\partial_z^\alpha a(z)| \leq C_\alpha \langle z \rangle^{m-|\alpha|} \ , \ \langle z \rangle = (1+|z|^2)^{1/2}.$$

Properties:

- $\Gamma_\rho^m(\mathbb{R}^{2n})$ is a (complex) vector space
- If $a \in \Gamma_\rho^m(\mathbb{R}^{2n})$ then $\partial_z^\alpha a \in \Gamma_\rho^{m-|\alpha|}(\mathbb{R}^{2n})$
- If $a \in \Gamma_\rho^m(\mathbb{R}^{2n})$ and $b \in \Gamma_\rho^{m'}(\mathbb{R}^{2n})$ then $ab \in \Gamma_\rho^{m+m'}(\mathbb{R}^{2n})$
- $\Gamma_\rho^{-\infty}(\mathbb{R}^{2n}) = \cap_{m \in \mathbb{R}} \Gamma_\rho^m(\mathbb{R}^{2n}) = \mathcal{S}(\mathbb{R}^{2n})$

Definition

Let $(a_j)_j$ be such that $a_j \in \Gamma_\rho^{m_j}(\mathbb{R}^{2n})$ such that $m_j \geq m_{j+1}$ and $\lim_{j\to\infty} m_j = -\infty$. Let $a \in C^\infty(\mathbb{R}^{2n})$. If or every integer $r \geq 2$ we have $a - \sum_{j=1}^{r-1} a_j \in \Gamma_\rho^M(\mathbb{R}^{2n})$ with $M = \max_{j\geq r} m_j$ we write $a \sim \sum_{j=1}^{\infty} a_j$.

We have:

Theorem

Let $A_{\mathrm{BJ}} = Op_{\mathrm{BJ}}(a)$ with $a \in \Gamma_\rho^m(\mathbb{R}^{2n})$. Then A_{BJ} is a Weyl operator $A_{\mathrm{W}} = Op_{\mathrm{W}}(b)$ with symbol $b \in \Gamma_\rho^m(\mathbb{R}^{2n})$ having the asymptotic expansion

$$b(x,p) \sim \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha| \text{ even}}} \frac{1}{\alpha!(|\alpha|+1)} \left(\frac{i\hbar}{2}\right)^{|\alpha|} \partial_x^\alpha \partial_p^\alpha a(x,p)$$

and we have $a - b \in \Gamma_\rho^{m-2\rho}(\mathbb{R}^{2n})$.

Conversely:

Theorem

Let $A_{\mathrm{W}} = Op_{\mathrm{W}}(a)$ *with* $a \in \Gamma_\rho^m(\mathbb{R}^{2n})$. *Let* $b \in \Gamma_\rho^m(\mathbb{R}^{2n})$ *with the asymptotic expansion*

$$b(x,p) \sim \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha| \text{ even}}} \frac{c_\alpha}{\alpha!}\left(\frac{i\hbar}{2}\right)^{|\alpha|} \partial_x^\alpha \partial_p^\alpha a(x,p)$$

where the c_α *are the coefficients appearing in the series* $\sum_{\alpha\in\mathbb{N}^n} \frac{c_\alpha}{\alpha!}$ *for the formal reciprocal of* $\sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha| \text{ even}}} \frac{1}{\alpha!(|\alpha|+1)}$. *Then* $B_{\mathrm{BJ}} = Op_{\mathrm{BJ}}(b)$ *is such that* $B_{\mathrm{BJ}} = A_{\mathrm{W}} + R$ *where* R *is anoperator with kernel in* $\mathcal{S}(\mathbb{R}^{2n})$.

Both theorems are proven in: E. Cordero, M. de Gosson, F. Nicola, *Trans. Amer. Math. Soc.* (Series B4), 94–109 (2017). The explicit form of the coefficients c_α is given (very complicated expressions).

Sobolev–Shubin Spaces

Definition

For $s \in \mathbb{R}$ the space $Q^s(\mathbb{R}^n)$ consists of all $\psi \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$Q^s(\mathbb{R}^n) = L^2_s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n).$$

The norm on $Q^s(\mathbb{R}^n)$ is defined by $||\psi||_{Q^s} = ||L_s\psi||_{L^2}$ where

$$L_s\psi(x) = \left(\tfrac{1}{2\pi\hbar}\right)^{n/2} \int e^{\frac{i}{\hbar}px} \langle z \rangle^{s/2} \widehat{\psi}(p) d^n p.$$

The space $Q^s(\mathbb{R}^n)$ can be equipped with an inner product making it into a Hilbert space.
The Sobolev–Shubin spaces are particular cases of **Feichtinger's modulation spaces**; in fact

$$Q^s(\mathbb{R}^n) = M^2_s(\mathbb{R}^{2n}).$$

A Continuity Result

Theorem

Let $a \in \Gamma_{\rho}^{m}(\mathbb{R}^{n})$. *(i) Then the Born–Jordan operator* $\widehat{A}_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a)$ *is a continuous operator*

$$\widehat{A}_{\mathrm{BJ}} : Q^{s}(\mathbb{R}^{n}) \longrightarrow Q^{s-m}(\mathbb{R}^{n}).$$

(ii) Let $a \in \Gamma_{\rho}^{0}(\mathbb{R}^{2n})$. *Then* $\widehat{A}_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a)$ *is a bounded operator on* $L^{2}(\mathbb{R}^{n})$.

Proof.

(i) Since $a \in \Gamma_{\rho}^{m}(\mathbb{R}^{n})$ we can write $\widehat{A}_{\mathrm{BJ}} = \mathrm{Op}_{\tau}(a_{\tau})$ with $a_{\tau} \in \Gamma_{\rho}^{m}(\mathbb{R}^{n})$. The result follows from the fact that $\mathrm{Op}_{\tau}(a_{\tau}) : Q^{s}(\mathbb{R}^{n}) \longrightarrow Q^{s-m}(\mathbb{R}^{n})$.
(ii) We have $Q^{0}(\mathbb{R}^{n}) = L^{2}(\mathbb{R}^{n})$. □

A Modulation Space

Definition

The modulation space $M^q(\mathbb{R}^n)$ consists of all $\psi \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathrm{Wig}(\psi, \phi) \in L^q(\mathbb{R}^{2n})$ for one (and hence every) $\phi \in \mathcal{S}(\mathbb{R}^n)$. The topology of $M^q(\mathbb{R}^n)$ is defined by any of the norms

$$||\psi||_{\phi, M^q} = ||W(\psi, \phi)||_{L^q_s}$$

$M^q(\mathbb{R}^n)$ is a Banach space for the topology thus defined by the norm $||\cdot||_{\phi, M^q_s}$ and $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of each of the modulation spaces $M^q_s(\mathbb{R}^n)$.

Theorem

Let $a \in C_b^{2n+1}(\mathbb{R}^{2n})$. The operator $\widehat{A}_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a)$ is bounded on $M^q(\mathbb{R}^n)$ for every $q \geq 1$.

The division problem

Recall that we have written $\widehat{A}_{\mathrm{W}} = \mathrm{Op}_{\mathrm{W}}(a)$ and $\widehat{A}_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a)$ as

$$\widehat{A}_{\mathrm{W}}\psi = \left(\tfrac{1}{2\pi\hbar}\right)^{n} \int_{\mathbb{R}^{2n}} a_{\sigma}(z)\widehat{T}(z)\psi dz \tag{25}$$

$$\widehat{A}_{\mathrm{BJ}}\psi = \left(\tfrac{1}{2\pi\hbar}\right)^{n} \int_{\mathbb{R}^{2n}} a_{\sigma}(z)\Theta(z)\widehat{T}(z)\psi dz$$

where $\psi \in \mathcal{S}(\mathbb{R}^n)$ and the integrals are to be understood in the distributional sense; here $\widehat{T}(z_0) = e^{-i(x_0\widehat{p} - p_0\widehat{x})/\hbar}$, $z_0 = (x_0, p_0)$, is the Heisenberg operator, $a_{\sigma}(z) = a_{\sigma}(x, p) = Fa(p, -x)$, with $z = (x, p)$, is the symplectic Fourier transform of a, and Θ is Cohen's kernel function, defined by

$$\Theta(z) = \frac{\sin(px/2\hbar)}{px/2\hbar}.$$

We have $\Theta(z_0) = 0$ if and only if $p_0 x_0 = 2N\pi\hbar$ for $N \in \mathbb{Z}$, $N \neq 0$.

It follows that the correspondence $a \longmapsto \mathrm{Op}_{\mathrm{BJ}}(a)$ is not injective: we have $\mathrm{Op}_{\mathrm{BJ}}(a + a_0) = \mathrm{Op}_{\mathrm{BJ}}(a)$ for every

$$a_0(z) = e^{-\frac{i}{\hbar}\sigma(z,z_0)} \text{ with } p_0x_0 = 2N\pi\hbar,\ N \in \mathbb{Z}, N \neq 0.$$

In fact, the symplectic Fourier transform of a_0 is

$$F_\sigma a_0(z) = (2\pi\hbar)^n \delta(z - z_0)$$

so that using the harmonic represenation of $\mathrm{Op}_{\mathrm{BJ}}(a_0)$ we have

$$\mathrm{Op}_{\mathrm{BJ}}(a_0) = \int \delta(z - z_0) \frac{\sin(px/2\hbar)}{px/2\hbar} \widehat{T}(z) dz = 0.$$

A Reduction Result

Theorem

Let $\widehat{A}_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a)$ with $a \in \Gamma_\rho^m(\mathbb{R}^{2n})$. (i) For every $\tau \in \mathbb{R}$ there exists $a_\tau \in \Gamma_\rho^m(\mathbb{R}^{2n})$ such that $\widehat{A}_{\mathrm{BJ}} = \mathrm{Op}_\tau(a_\tau)$. (ii) In particular, every Born–Jordan operator with symbol $a \in \Gamma_\rho^m(\mathbb{R}^{2n})$ is a Weyl operator with symbol $b \in \Gamma_\rho^m(\mathbb{R}^{2n})$.

Proof.

See M. de Gosson: Born–Jordan Quantization , Springer, 2016. □

Example

We have

$$\mathrm{Op}_{\mathrm{BJ}}(x^2p^2) = \tfrac{1}{3}(\widehat{x}^2\widehat{p}^2 + \widehat{p}\widehat{x}\widehat{p} + \widehat{p}^2\widehat{x}^2) = \mathrm{Op}_{\mathrm{W}}(\tfrac{4}{3}x^2p^2).$$

This property reduces the proof of many continuity results for BJ operators to the Weyl (or Shubin) case.

We have the following general result:

Theorem

The equation $a * \Theta_\sigma = b$ *admits a solution* $a \in \mathcal{S}'(\mathbb{R}^{2n})$, *for every* $b \in \mathcal{S}'(\mathbb{R}^{2n})$.

From the previous theorem, we obtain at once the following result.

Corollary

For every $b \in \mathcal{S}'(\mathbb{R}^{2n})$ *there exists a symbol* $a \in \mathcal{S}'(\mathbb{R}^{2n})$ *such that* $\mathrm{Op}_{\mathrm{BJ}}(a) = \mathrm{Op}_{\mathrm{W}}(b)$. *Hence, every linear continuous operator* $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ *can be written in Born-Jordan form, i.e. there exists a symbol* $a \in \mathcal{S}'(\mathbb{R}^{2n})$ *such that* $A = \mathrm{Op}_{\mathrm{BJ}}(a)$.

Detailed proofs are given in: E. Cordero, M. de Gosson, and F. Nicola, On the Invertibility of Born–Jordan Quantization, *J. Math. Pures Appl.* 05(4) 537–557 (2016)

The Paley–Wiener Theorem

The most popular version of this theorem says that the Fourier transform of a compactly supported distribution is an entire analytic function. A better statement is:

Theorem

Let $a \in \mathcal{E}'(\mathbb{R}^{2n})$ *have support* $\operatorname{supp}(a) \subset B^{2n}(r)$*. (i) The Fourier transform* Fa *can be extended into an entire analytic function on* $\mathbb{C}^{2n}$ *and there exists constants* $C > 0$*,* $N > 0$*, such that*

$$|Fa(\zeta)| \leq C(1 + |\zeta|)^N e^{\frac{r}{\hbar}|\operatorname{Im}\zeta|}; \tag{26}$$

where $\zeta = (\zeta_1, ..., \zeta_{2n})$ *and* $|\operatorname{Im}\zeta|^2 = |\operatorname{Im}\zeta_1|^2 + \cdots + |\operatorname{Im}\zeta_{2n}|^2$*. (ii) Every entire analytic function* a *on* $\mathbb{C}^{2n}$ *satisfying an estimate of the type (26) is the Fourier transform of some* $a \in \mathcal{S}'(\mathbb{R}^{2n})$ *such that* $\operatorname{supp}(a) \subset B^{2n}(r)$*.*

(see *e.g.* L. Hörmander, *The analysis of linear partial differential operators*, Vol. I, Springer-Verlag, 1981)

Paley–Wiener's theorem motivates the introduction of the following set of spaces:

Definition

For $r \geq 0$ we denote by $A_r(2n)$ the subspace of $\mathcal{S}'(\mathbb{R}^{2n})$ consisting of all tempered distributions a whose symplectic Fourier transform a_σ has support $\operatorname{supp}(a_\sigma) \subset B^{2n}(r)$. Equivalently, a satisfies an estimate

$$|a(\zeta)| \leq C(1+|\zeta|)^N e^{\frac{r}{\hbar}|\operatorname{Im}\zeta|} \tag{27}$$

for some constants $C > 0$, $N > 0$.

Obviously $A_0(2) = \mathbb{C}[x,p]$, the space of polynomials in the real variables x and p, since $a \in A_0(2)$ if and only if a is polynomially bounded, and hence a polynomial because $\operatorname{supp}(a_\sigma) = \{0\}$.
More generally, $A_0(2n)$ is the space of polynomials in the variables $x_1, ..., x_n$ and $p_1, ..., p_n$.

We have proven that:

Theorem

The linear mapping $\mathcal{S}'(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ *defined by*

$$a \longmapsto a_\sigma \Theta \quad , \quad \Theta(x,p) = \frac{\sin(px/2\hbar)}{px/2\hbar} \tag{28}$$

restricts to an automorphism of $A_r(2n)$ *if and only if*

$$0 \leq r < \sqrt{4\pi\hbar}. \tag{29}$$

Proof.

Uses the theory of division of distributions. See E. Cordero, M. de Gosson, F. Nicola: On the Invertibility of Born-Jordan Quantization, *J. Math. Pures Appl.* (2015). □

In particular, taking $r = 0$, every polynomial in $\widehat{x}, \widehat{p}$ is the Born–Jordan quantization of a unique polynomial in x, p (which was not obvious from the beginning).

Acknowledgements: This work has been supported by the Grants P 27773 and P 33447 of the Austrian Research Foundation FWF (Fonds zur Förderung der wissenschaftlichen Forschung).

.THANK YOU FOR YOUR KIND ATTENTION!!!