NTNU | Norwegian University of Science and Technology

DECOUPLING FOR SCHATTEN CLASS OPERATORS IN THE SETTING OF QUANTUM HARMONIC ANALYSIS

Workshop on Quantum Harmonic Analysis - Hannover

Helge Jørgen Samuelsen

August 06, 2024

Let $\mathscr{S}(\mathbb{R}^{2d})$ denote the Schwartz class of rapidly decaying smooth functions, and $\mathscr{S}'(\mathbb{R}^{2d})$ its dual space of tempered distributions.

Symplectic Fourier transform

For $\psi \in \mathscr{S}(\mathbb{R}^{2d})$ we denote the symplectic Fourier transform of ψ at a point $\zeta \in \mathbb{R}^{2d}$ by

$$\mathcal{F}_\sigma(\psi)(\zeta) = \int_{\mathbb{R}^{2d}} e^{-2\pi i \sigma(\zeta, z)} \psi(z)\, dz,$$

where σ is the symplectic form on $\mathbb{R}^{2d}$. For $\tau \in \mathscr{S}'(\mathbb{R}^{2d})$, we extend the sympletic Fourier transform to $\mathscr{S}'(\mathbb{R}^{2d})$ through,

$$\langle \mathcal{F}_\sigma(\tau), \psi \rangle_{\mathscr{S}', \mathscr{S}} = \langle \tau, \mathcal{F}_\sigma(\psi) \rangle_{\mathscr{S}', \mathscr{S}}, \quad \forall \psi \in \mathscr{S}(\mathbb{R}^{2d}),$$

where a sesquilinear dual pairing is used.

Notation and Background - Classical Decoupling

Let us introduces the classical decoupling constant.

Definition

Let $\Omega \subset \mathbb{R}^{2d}$, and $\mathcal{P}_\Omega$ be a partition of Ω. We define the classical decoupling constant $\mathcal{D}^{\mathscr{C}}_{p,q}(\mathcal{P}_\Omega)$ to be the smallest constant for which the inequality

$$\left\| \sum_{\theta \in \mathcal{P}_\Omega} f_\theta \right\|_{L^p(\mathbb{R}^{2d})} \leq \mathcal{D}^{\mathscr{C}}_{p,q}(\mathcal{P}_\Omega) \left(\sum_{\theta \in \mathcal{P}_\Omega} \| f_\theta \|^q_{L^p(\mathbb{R}^{2d})} \right)^{\frac{1}{q}},$$

holds for any collection of functions with supp $\mathcal{F}_\sigma(f_\theta) \subseteq \theta$.

The goal of decoupling is to achieve asymptotic bounds for the decoupling constant when considering finer and finer partitions, i.e. $\#\mathcal{P}_\Omega \to \infty$.
It is mostly $q = 2$ which is considered in the literature.

Notation and Background - Basic estimates

Square function estimates

By Minkowski's inequality for $p > q$, it follows that

$$\left\| \left(\sum_{\theta \in \mathcal{P}_\Omega} |f_\theta|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^{2d})} \leq \left(\sum_{\theta \in \mathcal{P}_\Omega} \|f_\theta\|_{L^p(\mathbb{R}^{2d})}^q \right)^{\frac{1}{q}}.$$

In particular, for $p > 2$, every square function estimate gives rise to a decoupling result for $q = 2$. We may therefore think of decoupling as a weaker square function estimate.

Trivial decoupling constant for finite partitions

Assume the partition $\mathcal{P}_\Omega$ consist of N elements. Then Hölder's inequality yields

$$\left\| \sum_{\theta \in \mathcal{P}_\Omega} f_\theta \right\|_{L^p(\mathbb{R}^{2d})} \leq \sum_{\theta \in \mathcal{P}_\Omega} \|f_\theta\|_{L^p(\mathbb{R}^{2d})} \leq N^{\frac{1}{q'}} \left(\sum_{\theta \in \mathcal{P}_\Omega} \|f_\theta\|_{L^p(\mathbb{R}^{2d})}^q \right)^{\frac{1}{q}}$$

Classical results - Title of frame

Consider the truncated paraboloid

$$\mathbb{P}^{2d-1} := \{(\xi, |\xi|^2) \in \mathbb{R}^{2d} : \xi \in \mathbb{R}^{2d-1}, |\xi_i| \leq 1 \text{ for } 1 \leq i \leq 2d-1\},$$

and cover the δ-neighbourhood by

$$\theta = \{(\xi, |\xi|^2 + t) : \xi \in C_\theta, |t| < \delta\},$$

where $\{C_\theta\}$ is a partition of $[-1,1]^{2d-1}$ by cubes of side length $\delta^{\frac{1}{2}}$.

Theorem (Bourgain-Demeter '15)

For $0 < \delta < 1$ and any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that if $2 \leq p \leq (4d+2)/(2d-1)$, then

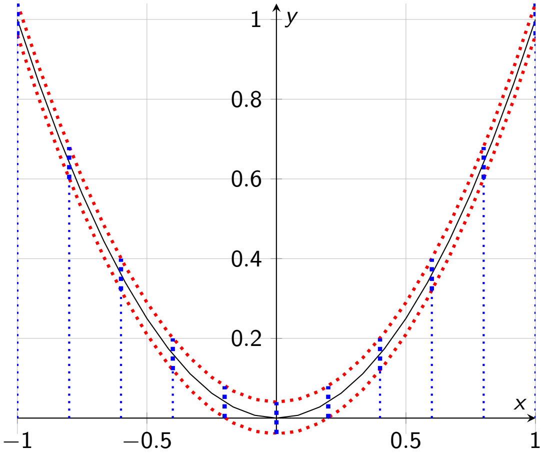
$$\mathcal{D}^{\mathscr{C}}_{p,2}(\mathcal{P}_{\mathcal{N}_{\mathbb{P}^{2d-1}}(\delta)}) \leq C_\varepsilon \delta^{-\varepsilon},$$

while if $p \geq (4d+2)/(2d-1)$, then

$$\mathcal{D}^{\mathscr{C}}_{p,2}(\mathcal{P}_{\mathcal{N}_{\mathbb{P}^{2d-1}}(\delta)}) \leq C_\varepsilon \delta^{-\varepsilon + \frac{2d+1}{2p} - \frac{2d-1}{4}}.$$

The proved the result for compact hypersurfaces with non-vanishing Gaussian curvature.

Classical results - Figure of the δ-neighbourhood



Classical results - Decoupling for the moment curve

Consider the moment curve $\Gamma : [0,1] \to \mathbb{R}^{2d}$ given by

$$\Gamma(t) = (t, t^2, \dots, t^{2d-1}, t^{2d}).$$

For each $0 < \delta < 1$, let $\mathcal{P}(\delta)$ be a dyadic partition

$$[0,1] = \bigcup_{J \in \mathcal{P}(\delta)} J$$

where each interval $|J| = 2^{\lceil \log_2(\delta) \rceil}$. Moreover, for each interval J let c_J be the centre of J and denote by $\mathcal{U}_J$ the parallelepiped centred at $\Gamma(c_J)$ of dimensions $|J| \times |J|^2 \times \dots \times |J|^{2d}$ where side number j is parallel to $\Gamma^{(j)}(c_J)$.

Theorem (Bourgain-Demeter-Guth '16)

For each $d \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a finite constant $C_{d,\varepsilon} > 0$ such that

$$\mathcal{D}^{\mathscr{C}}_{2d(2d+1),2}(\mathcal{P}(\delta)) \leq C_{d,\varepsilon} \delta^{-\varepsilon},$$

for every $0 < \delta < 1$.

Classical results - Application to exponential sums

Lemma (Reverse Hölder)

Let $|\mathcal{P}_\Omega| = N \in \mathbb{N}$. *Then for each* $1 \leq p, q \leq \infty$ *and* $a \in \ell^q(\{1, \ldots, N\})$ *and* $R \gtrsim \max|\theta|^{-1}$,

$$\left(\fint_{B_R(0)} \left| \sum_{\theta \in \mathcal{P}_\Omega} a_\theta e^{2\pi i \sigma(x, \xi_\theta)} \right|^p dx \right)^{\frac{1}{p}} \lesssim \mathcal{D}^{\mathscr{C}}_{p,q}(\mathcal{P}_\Omega) \|a\|_{\ell^q},$$

where $\xi_\theta \in \theta$.

Corollary (Application to the moment curve)

For each $1 \leq i \leq N$, *let* $t_i \in ((i-1)/N, i/N]$. *For each* $R \gtrsim N^{2d}$, *and every* $a \in \ell^2(\{1, \ldots, N\})$ *and every* $p \geq 2$ *there exists* $C_\varepsilon > 0$ *such that*

$$\left(\fint_{B_R(0)} \left| \sum_{l=1}^{N} a_l e^{2\pi i \sigma(x, \Gamma(t_l))} \right|^p dx \right)^{\frac{1}{p}} \leq C_\varepsilon \left(N^\varepsilon + N^{\frac{1}{2} - \frac{2d(2d+1)}{2p} + \varepsilon} \right) \|a\|_{\ell^2}$$

Classical results - Sketch of proof for reverse Hölder

Proof.
Let $\psi \in \mathscr{S}(\mathbb{R}^{2d})$ be such that $|\psi| \geq 1$ on $B_1(0)$ and $\mathcal{F}_\sigma(\psi)$ is supported on $B_1(0)$. Then the function

$$F_\theta(z) = a_\theta \psi_R(z) e^{2\pi i \sigma(z,\xi_\theta)}$$

has its symplectic Fourier transform support on θ. Here $\psi_R(z) = \psi(z/R)$.
Decoupling then gives

$$\left(\int_{B_R(0)} \left| \sum_{\theta \in \mathcal{P}_\Omega} a_\theta e^{2\pi i \sigma(z,\xi_\theta)} \right|^p dz \right)^{\frac{1}{p}} \leq \left\| \sum_{\theta \in \mathcal{P}_\Omega} F_\theta \right\|_{L^p} \leq \mathcal{D}^{\mathscr{C}}_{p,q}(\mathcal{P}_\Omega) \left(\sum_{\theta \in \mathcal{P}_\Omega} \|a_\theta \psi_R\|^q_{L^p} \right)^{\frac{1}{q}}$$
$$= R^{\frac{2d}{p}} \|\psi\|_{L^p} \mathcal{D}^{\mathscr{C}}_{p,q}(\mathcal{P}_\Omega) \|a\|_{\ell^q}.$$

The proof is concluded by recalling that $|B_r(0)| \approx R^{2d}$. □

Time-frequency Analysis

Let ρ denote the symmetric time-frequency shift given by

$$\rho(x,\xi)f(t) = e^{-\pi i x\cdot\xi} e^{2\pi i \xi\cdot t} f(t-x), \quad (x,\xi) \in \mathbb{R}^{2d},\ f \in L^2(\mathbb{R}^d).$$

Associated to ρ is the cross-ambiguity function. For $f, g \in L^2(\mathbb{R}^d)$, the cross-ambiguity function of f and g is defined at a point $z \in \mathbb{R}^{2d}$ by

$$\mathcal{A}(f,g)(z) = \langle f, \rho(z)g\rangle_{L^2} \in L^2(\mathbb{R}^{2d}).$$

$$\mathcal{W}(f,g)(z) = \mathcal{F}_\sigma\left(\mathcal{A}(f,g)\right)(z).$$

Schatten Class

Let $\mathcal{K}$ denote the compact operators on $L^2(\mathbb{R}^d)$. By the singular value decomposition, for each $T \in \mathcal{K}$ we can write it as

$$T = \sum_{n\in\mathbb{N}} s_n(T)\varphi_n \otimes \psi_n,$$

where $\{s_n(T)\}$ are the singular values of T, and $\{\varphi\}, \{\psi_n\}$ are two orthonomal families in L^2. We denote the Schatten p-class of compact operators on $L^2(\mathbb{R}^d)$ by

$$\mathcal{S}^p = \{T \in \mathcal{K} : \{s_n(T)\} \in \ell^p\},$$

and equip it with the norm

$$\|T\|_{\mathcal{S}^p} = \left(\sum_{n\in\mathbb{N}} |s_n(T)|^p\right)^{\frac{1}{p}},$$

which turn it into a Banach space.

Weyl quantization

For each $\tau \in \mathscr{S}'(\mathbb{R}^{2d})$, we can associate it to a continuous linear operator from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$, denoted L_τ, by the cross-Wigner distribution,

$$\langle L_\tau f, g \rangle_{\mathscr{S}',\mathscr{S}} = \langle \tau, \mathcal{W}(g,f) \rangle_{\mathscr{S}',\mathscr{S}},$$

for all $f, g \in \mathscr{S}(\mathbb{R}^d)$. Here we use a sesquilinear dual pairing. This is known as the Weyl quantization of τ, and τ is called the Weyl symbol of L_τ. In fact, every continuous linear operator from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ has a Weyl symbol.

Schwartz and tempered distribution classes of operators

Define the Schwartz class of operators by

$$\mathfrak{S} = \{L_\varphi : \varphi \in \mathscr{S}(\mathbb{R}^{2d})\} \subseteq \mathcal{S}^1,$$

and its dual space

$$\mathfrak{S}' \cong \mathcal{L}\left(\mathscr{S}(\mathbb{R}^d), \mathscr{S}'(\mathbb{R}^d)\right).$$

Convolutions

We then define an operator-operator convolution by

$$T \star S(z) := \mathrm{tr}(T\alpha_z(PSP)),$$

and a function-operator convolution by

$$f \star T = T \star f := \int_{\mathbb{R}^{2d}} f(z)\alpha_z(T)\,dz,$$

where the integral is considered weakly. Here $\alpha_z(T) = \rho(z)T\rho(-z)$ and $Pf(t) = f(-t)$ is the Parity operator.

Theorem (Werner-Young)

Let $1 \leq p, q, r \leq \infty$ be such that $1 + r^{-1} = p^{-1} + q^{-1}$. If $f \in L^p(\mathbb{R}^{2d})$, $T \in \mathcal{S}^p$ and $S \in \mathcal{S}^q$, then $f \star S \in \mathcal{S}^r$ and $T \star S \in L^r(\mathbb{R}^{2d})$. Moreover, there are the norm bounds

$$\|f \star S\|_{\mathcal{S}^r} \leq \|f\|_{L^p}\|S\|_{\mathcal{S}^q},$$
$$\|T \star S\|_{L^r} \leq \|T\|_{\mathcal{S}^p}\|S\|_{\mathcal{S}^q}.$$

Fourier-Wigner transform

For $T \in \mathcal{S}^1$, and $z \in \mathbb{R}^{2d}$, we define the Fourier-Wigner (or Fourier-Weyl) transform as

$$\mathcal{F}_W(T)(z) = \mathrm{tr}(T\rho(-z)) \in C_0(\mathbb{R}^{2d}).$$

It also extends to an isomorphism from $\mathcal{S}^2$ to $L^2(\mathbb{R}^{2d})$ due to a result of Pool.
Since $\mathfrak{S} \subseteq \mathcal{S}^1$, the Fourier-Wigner is well-defined and can be extended by duality to $\mathfrak{S}'$.

Theorem (Keyl-Kiukas-Werner '16)

Let $S \in \mathfrak{S}$, $\psi \in \mathscr{S}(\mathbb{R}^{2d})$, $\tau \in \mathscr{S}'(\mathbb{R}^{2d})$ and $A \in \mathfrak{S}'$. Then relations

$$\mathcal{F}_\sigma(S \star A) = \mathcal{F}_W(S)\mathcal{F}_W(A), \qquad \mathcal{F}_W(\psi \star A) = \mathcal{F}_\sigma(\psi)\mathcal{F}_W(A),$$
$$\mathcal{F}_\sigma(\psi * \tau) = \mathcal{F}_\sigma(\psi)\mathcal{F}_\sigma(\tau), \qquad \mathcal{F}_W(S \star \tau) = \mathcal{F}_W(S)\mathcal{F}_\sigma(\tau),$$

hold.

Extending decoupling to operators

We can now extend decoupling to operators with the following definition.

Definition

Let $\Omega \subset \mathbb{R}^{2d}$, and $\mathcal{P}_\Omega$ be a partition of Ω. We define the quantum decoupling constant $\mathcal{D}^{\mathscr{Q}}_{p,q}(\mathcal{P}_\Omega)$ to be the smallest constant for which the inequality

$$\left\| \sum_{\theta \in \mathcal{P}_\Omega} T_\theta \right\|_{\mathcal{S}^p} \leq \mathcal{D}^{\mathscr{Q}}_{p,q}(\mathcal{P}_\Omega) \left(\sum_{\theta \in \mathcal{P}_\Omega} \| T_\theta \|^q_{\mathcal{S}^p} \right)^{\frac{1}{q}},$$

holds for any collection of operators $\{T_\theta\}_{\theta \in \mathcal{P}_\Omega} \subset \mathcal{L}(\mathscr{S}(\mathbb{R}^d); \mathscr{S}'(\mathbb{R}^d))$ with $\operatorname{supp}(\mathcal{F}_W(T_\theta)) \subseteq \theta$.

Equivalence of classical and quantum decoupling constants

We are able to show the following equivalence of decoupling under the assumption of Ω being a bounded set.

Theorem

Let $\Omega \subset \mathbb{R}^{2d}$ be a bounded set, and let $\mathcal{P}_\Omega$ be a partition of Ω. Then there exists $C = C(\Omega) \geq 1$ such that

$$C^{-1}\mathcal{D}^{\mathscr{C}}_{p,q}(\mathcal{P}_\Omega) \leq \mathcal{D}^{\mathscr{Q}}_{p,q}(\mathcal{P}_\Omega) \leq C\,\mathcal{D}^{\mathscr{C}}_{p,q}(\mathcal{P}_\Omega)$$

holds for any $1 \leq p, q \leq \infty$.

Remark

The constant C in the theorem is independent of p, q and the partition $\mathcal{P}_\Omega$.

Main results - Quantum decoupling for the Paraboloid

Consider the truncated paraboloid

$$\mathbb{P}^{2d-1} := \{(\xi, |\xi|^2) \in \mathbb{R}^{2d} : \xi \in \mathbb{R}^{2d-1}, |\xi_i| \leq 1 \text{ for } 1 \leq i \leq 2d-1\},$$

and again cover the δ-neighbourhood of $\mathbb{P}^{2d-1}$ by

$$\theta = \{(\xi, |\xi|^2 + t) : \xi \in C_\theta, |t| < \delta\},$$

where $\{C_\theta\}$ is a partition of $[-1, 1]^{2d-1}$ by cubes of side length $\delta^{\frac{1}{2}}$.

$\ell^2\mathcal{S}^p$-quantum decoupling

For $0 < \delta < 1$ and any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that if $2 \leq p \leq (4d+2)/(2d-1)$, then

$$\mathcal{D}^{\mathscr{Q}}_{p,2}(\mathcal{P}_{\mathcal{N}_{\mathbb{P}^{2d-1}}(\delta)}) \leq C_\varepsilon \delta^{-\varepsilon},$$

while if $p \geq (4d+2)/(2d-1)$, then

$$\mathcal{D}^{\mathscr{Q}}_{p,2}(\mathcal{P}_{\mathcal{N}_{\mathbb{P}^{2d-1}}(\delta)}) \leq C_\varepsilon \delta^{-\varepsilon + \frac{2d+1}{2p} - \frac{2d-1}{4}}.$$

Main results - A convolution result

Two Convolution results

Proposition

Let Ω be a bounded set in $\mathbb{R}^{2d}$. There exist L^2-normalised $g, h \in \mathscr{S}(\mathbb{R}^d)$ and $B \in \mathfrak{S}$ such that:

i) *If $\tau \in \mathscr{S}'(\mathbb{R}^{2d})$ and $\mathcal{F}_\sigma(\tau)$ is supported on Ω, then $\tau = \tau \star (g \otimes h) \star B$.*

ii) *If $T \in \mathfrak{S}'$ and $\mathcal{F}_W(T)$ is supported on Ω, then $T = T \star (g \otimes h) \star B$.*

Corollary

Let Ω be a bounded set, and $1 \leq p \leq \infty$. There exists L^2-normalised $g, h \in \mathscr{S}(\mathbb{R}^d)$ and $C = C(\Omega) > 0$ such that

i) *If $\tau \in \mathscr{S}'(\mathbb{R}^{2d})$ and $\mathcal{F}_\sigma(\tau)$ is supported on Ω, then*

$$\|\tau\|_{L^p(\mathbb{R}^{2d})} \leq C(\Omega) \|\tau \star (g \otimes h)\|_{\mathcal{S}^p}.$$

ii) *If $T \in \mathfrak{S}'$ and $\mathcal{F}_W(T)$ is supported on Ω, then*

$$\|T\|_{\mathcal{S}^p} \leq C(\Omega) \|T \star (g \otimes h)\|_{L^p(\mathbb{R}^{2d})}.$$

Sketch of the proof - Proof of Convolution result

- There exist $z_0 \in \Omega$ and $R > 0$ such that $\Omega \subseteq B_R(z_0)$. Let $\Psi \in C_c^\infty(\mathbb{R}^{2d})$ be a smooth cut-off function such that $\Psi \equiv 1$ on $B_R(z_0)$ and supported on $B_{2R}(z_0)$. Moreover, let $g, h \in \mathscr{S}(\mathbb{R}^d)$ be such that $|\mathcal{A}(g,h)(z)| \geq \delta > 0$ on $B_{2R}(z_0)$.
- Then, for each $z \in \Omega$,

$$\Psi(z) = \frac{\mathcal{A}(g,h)(z)}{\mathcal{A}(g,h)(z)}\Psi(z) = \mathcal{F}_W(g \otimes h)(z)\mathcal{F}_W(B)(z),$$

where the operator B is given by

$$B = \mathcal{F}_W^{-1}\left(\frac{\Psi}{\mathcal{A}(g,h)}\right) = \int_{\mathbb{R}^{2d}} \frac{1}{\mathcal{A}(g,h)(z)}\Psi(z)\rho(z)\,dz.$$

- For each $T \in \mathfrak{S}'$ such that $\operatorname{supp} \mathcal{F}_W(T) \subseteq \Omega$, it follows that

$$\mathcal{F}_W(T) = \mathcal{F}_W(T)\Psi = \mathcal{F}_W(T)\mathcal{F}_W(g \otimes h)\mathcal{F}_W(B) = \mathcal{F}_W(T \star (g \otimes h) \star B),$$

and the result follows as $\mathcal{F}_W$ is an isomorphism from $\mathfrak{S}'$ to $\mathscr{S}'(\mathbb{R}^{2d})$.

Sketch of the proof - Proof of decoupling equivalence

- Let $\{T_\theta\}$ be a collection for which supp $\mathcal{F}_W(T_\theta) \subseteq \theta$. Then $T = \sum_\theta T_\theta$ has its Fourier-Wigner transform supported on Ω. Thus, by the corollary

$$\left\| \sum_{\theta \in \mathcal{P}_\Omega} T_\theta \right\|_{\mathcal{S}^p} \leq C(\Omega) \left\| \sum_{\theta \in \mathcal{P}_\Omega} T_\theta \star (g \otimes h) \right\|_{L^p(\mathbb{R}^{2d})} \leq C(\Omega) \mathcal{D}^{\mathscr{C}}_{p,q} \left(\sum_{\theta \in \mathcal{P}_\Omega} \| T_\theta \star (g \otimes h) \|^q_{L^p(\mathbb{R}^{2d})} \right)^{\frac{1}{q}},$$

as the Function $F_\theta = T_\theta \star (g \otimes h)$ has its symplectic Fourier transform supported on θ.

- Applying Werner-Young's convolution theorem gives

$$\left\| \sum_{\theta \in \mathcal{P}_\Omega} T_\theta \right\|_{\mathcal{S}^p} \leq C(\Omega) \mathcal{D}^{\mathscr{C}}_{p,q} \left(\sum_{\theta \in \mathcal{P}_\Omega} \| T_\theta \star (g \otimes h) \|^q_{L^p(\mathbb{R}^{2d})} \right)^{\frac{1}{q}} \leq C(\Omega) \mathcal{D}^{\mathscr{C}}_{p,q} \left(\sum_{\theta \in \mathcal{P}_\Omega} \| T_\theta \|^q_{\mathcal{S}^p} \right)^{\frac{1}{q}},$$

as g, h are L^2-normalised.

Thank you for your attention

NTNU | Norwegian University of Science and Technology