Problem Session

Problem 1 (Fulsche)

Assume that $f \in \mathrm{BUC}(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ is regular (meaning that $\mathcal{F}(g)(\xi) \neq 0$ for all $\xi \in \mathbb{R}$). Then we know that $g * f \in C_0(\mathbb{R})$ if and only if $f \in C_0(\mathbb{R})$. Now assume that g is not regular.

Question: Does $g * f \in C_0(\mathbb{R})$ imply that $f \in C_0(\mathbb{R}) + \mathrm{Ker}(h \mapsto g * h)$?

Comments: The other direction is clear. The answer probably depends on the set $S := \{\xi \in \mathbb{R} : \mathcal{F}(g)(\xi) = 0\}$. Speculation by Fulsche (?): YES if S is a subgroup of $\mathbb{R}$, NO if S is not a set of spectral synthesis.

Problem 2 (Fulsche)

Let (Ξ, m) be an abelian phase space. That is, Ξ is an lca group, $m\colon \Xi \times \Xi \to S^1$ satisfies

$$m(x+y,z)m(x,y) = m(x,y+z)m(y,z)$$

and $\sigma\colon \Xi \times \Xi \to S^1$, $\sigma(x,y) := \frac{m(x,y)}{m(y,x)}$ satisfies that $y \mapsto \sigma(\cdot, y)$ is an isomorphism between Ξ and $\widehat{\Xi}$. By Stone–von Neumann, there exists a unique irreducible projective representation ρ of Ξ such that $\rho(x)\rho(y) = m(x,y)\rho(x+y)$. This representation is automatically square integrable.

Question: Is ρ integrable?

Comments: If Ξ is of the form $G \times \widehat{G}$ and $m\big((x,\xi),(y,\eta)\big) = \xi(y)$, then ρ is integrable.

Problem 3 (Fulsche)

Question: Is there a notion of (compact) support for operators? Is there a QHA version of the Payley–Wiener theorem?

Comments: Maybe this question needs a bit more elaboration.

Problem 4 (Feichtinger)

Let $(h_n)_{n\in\mathbb{N}_0}$ be a system of Hermite functions and consider the operator

$$P_k\colon S_0 \to S_0, \quad P_k f := \sum_{n=0}^{k} \langle f, h_n \rangle h_n.$$

Question: Can we give a bound for $\|P_k\|_{S_0\to S_0}$?

Comments: The norm of $f \mapsto \langle f, h_n \rangle h_n$ is bounded independently of n.

Compare the following classical paper, showing that one has convergence in $L^p(R)$ for p close enough to $p = 2$, because otherwise these partial sum operators are not bounded!

R. Askey and S. Wainger. Mean convergence of expansions in Laguerre und Hermite series. *Amer. J. Math.*, 87(3):695–708, 1965.

Speculation by Fulsche (?): Might be related to the Berger–Coburn conjecture for Toeplitz operators on the Fock space. CITATION please: *?*

C. A. Berger and L. A. Coburn. Toeplitz operators on the Segal-Bargmann space. *Trans. Amer. Math. Soc.*, 301(2):813–829, 1987.

Problem 5 (Feichtinger)

It is well known that for any $g \in S_0(R)$ (or modulation space $M^1(R)$) one has: Given such g there exists constants a_0, b_0 such that one can assure that for any $a \in (0, a_0]$ and $b \in (0, b_0]$ the Gabor family arising from g with lattice $\Lambda = aZ \times bZ$ is a Gabor frame for $L^2(R)$.

Now it is well known that the fractional Fourier transform FT_α leaves $S_0(R)$ invariant, for any $\alpha \in R$.

Question: Given $g \in S_0$, do there exist $a_1, b_1 \in \mathbb{R}_+$ such that if $a \in (0, a_1]$ and $b \in (0, b_1]$, the family $\big(\pi(ak, bl)\mathcal{F}_{\alpha_{k,l}} g\big)_{(k,l)\in\mathbb{R}^{2d}}$ is a Gabor frame of fixed quality? Thus at each of the lattice points a random fractional FT of g can be used, and independent of the resulting family is a frame with joint frame bounds. In fact, one even would assume that the dual frames are well localized near the corresponding points in phase space and uniformly bounded in the norm of $S_0(R)$.

Comments: Corresponding results have been derived in the master thesis, but only for atoms g close to the classical Gauss function which is in fact invariant under the family of fractional Fourier transforms. In such a case it is more or less comparable to the standard perturbation of frames arguments:

A. Missbauer. Gabor Frames and the Fractional Fourier Transform. Master's thesis, University of Vienna, 2012.

Problem 6 (Dewage)

Let $\mathcal{A}^2$ denote the Bergman space of the unit ball $\mathbb{B}^n \subseteq \mathbb{C}^n$, $\alpha \in \mathbb{N}_0$ and $S \in \mathcal{L}(\mathcal{A}^2)$. We define $\varphi_\alpha(z) := \binom{n+\alpha}{\alpha}(1-|z|^2)^{n+1+\alpha}$, $\Phi := 1 \otimes 1$ and Φ_α such that $\Phi_\alpha * \Phi = \varphi_\alpha$. Moreover, let $B_\alpha(S) := S * \Phi_\alpha$.

Question: Is $\|(S * \Phi_\alpha) * \Phi\|_{\mathrm{op}} = \|B_\alpha(S) * \Phi\|_{\mathrm{op}}$ bounded independently of α?

Comments: $(S * \Phi_\alpha) * \Phi$ is equal to the Toeplitz operator $T_{B_\alpha(S)}$. Also note that

$$T_{B_\alpha(S)} * \Phi = B_\alpha(S) * \varphi_0 = B_0(S) * \varphi_\alpha,$$

which is bounded by $\|S\|_{\mathrm{op}}\|\varphi_\alpha\|_1 = \|S\|_{\mathrm{op}}$. Moreover, we know that $\|S * \Phi_\alpha\|_\infty$ is bounded independently of α if and only if S is a Toeplitz operator.

Problem 7 (Dewage)

Same notation as in the previous question.

Question: Is $\mathrm{BUC}(\mathbb{D}) * \mathcal{S}^1$ a closed set of operators?

Comments: This is motivated by the question whether for any operator S in the Toeplitz algebra the sequence of Toeplitz operators $(T_{S*\Phi_\alpha})_{\alpha\in\mathbb{N}_0}$ converges to S in operator norm as $\alpha \to \infty$.

Problem 8 (Toft)

Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}^{2d})$, $p \in (0, \infty]$ and denote by s^w_p the set of symbols such that the Weyl quantization $\mathrm{Op}^w(a)$ of a is in the Schatten class $\mathcal{S}^p$. Moreover, let φ and ψ be some Schwarz functions.

Question: Does $\varphi a \in s^w_p$, $(1-\psi)a \in s^w_\infty$ imply $a \in s^w_p$?

Comments: For $p = 1$ and $p = \infty$ this is true, so maybe interpolation could be useful.