Joint measurement of quasi-free observables in phase space

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Overview

- General concepts
  - Quantum observables and their joint measurability
  - The relevant case: phase space localisation
  - Implications of covariance
- Phase space setting
  - Quasi-free (covariant) observables
  - Joint measurability of several quasi-free observables
  - A necessary condition for quadratures

Origin of the joint measurability problem

- Goes back to the Uncertainty Principle (Heisenberg[1]):

  Some physical "observables" cannot be simultaneously measured (with arbitrary precision).

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

- Traditional mathematical setting:
  - Observables are self-adjoint operators, or collections of orthogonal projections (spectral theorem)
  - Observables are jointly measurable iff they commute.
- This setting is not sufficient for quantum information purposes!

1) **W. Heisenberg** Z. Phys. 43 172–198 (1927).

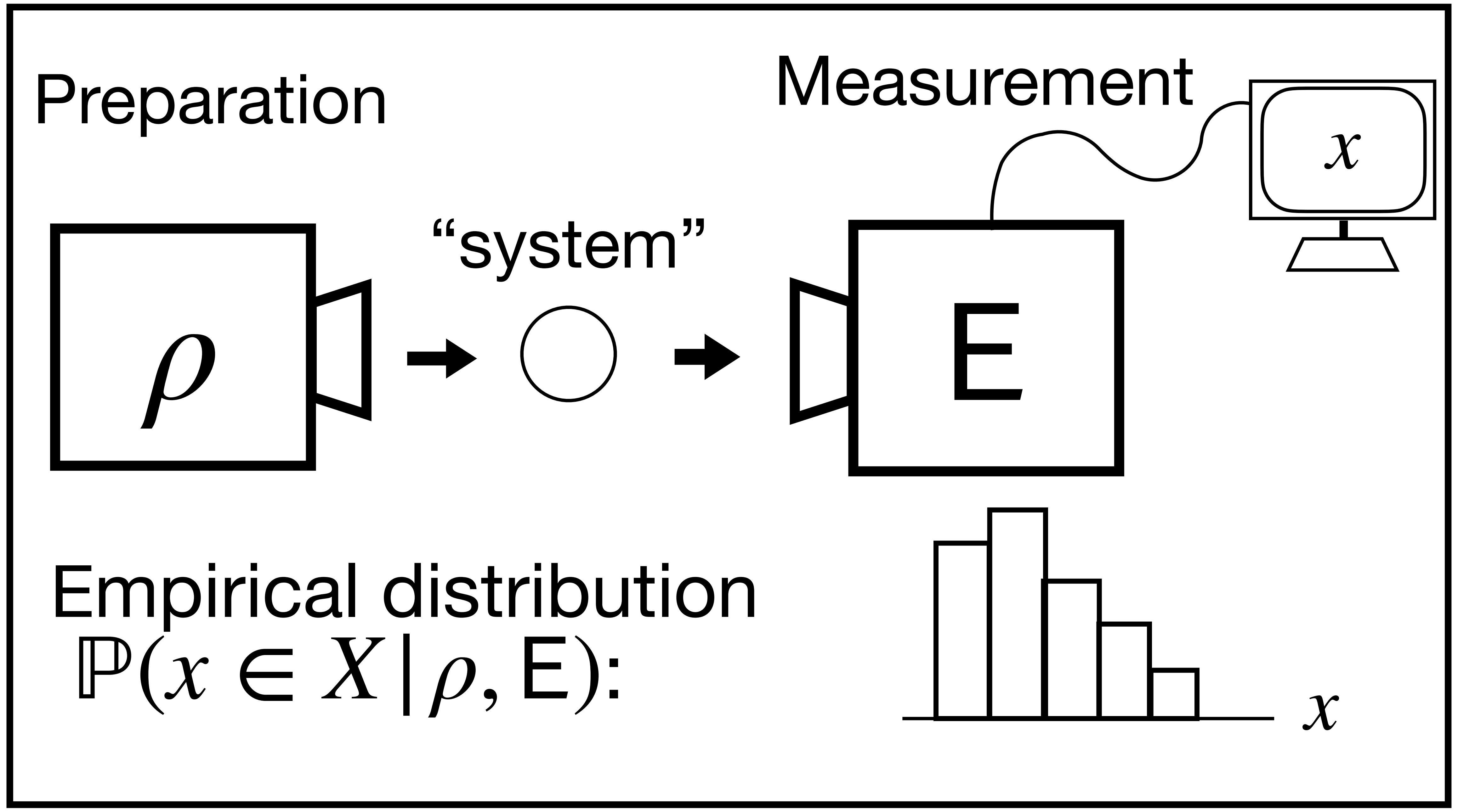
Modern formulation of quantum measurements

- Based on generalised observables — collections of positive operators, or "effect cones" in probabilistic theories (about physical systems).
- Originated[1] (in the 1960s) within the statistical "operational / empiricist" interpretation and axiomatisation of quantum mechanics[1,2].
- Developed subsequently in infinite-dimensional settings including CCR and quantum harmonic analysis[2].
- Currently (since ~ 10 years) used extensively in quantum information community, in finite-dimensional setting.

1) **G. Ludwig**, Deutung des Begriffs "physikalische Theorie" und axiomatische Grundlegung der Hilbertraumstruktur der Quantenmechanik durch Hauptsaetze des Messens, Lecture Notes in Physics 4, Springer 1970.
2) **R. Werner**, J. Math. Phys. 25 1404 (1984); **A. S. Holevo**, Probabilistic and statistical aspects of quantum theory, North-Holland Series in Statistics and Probability, Vol. 1, North-Holland 1982; **E. B. Davies**, Quantum Theory of Open Systems, Academic Press, 1976; **P. Busch, M. Grabowski, P. J. Lahti**, Operational quantum physics, LNPMGR, vol 31, Springer 1995.

Probabilistic framework of measurements

- Repeated preparation of a “system” defines a *state* ρ.
- *Observable* E with outcomes $x \in \mathscr{X}$ is defined by empirical probabilities $\mathbb{P}(x \in X \,|\, \rho, \mathsf{E})$ for $X \subset \mathscr{X}$.
- A probabilistic theory gives a *model* $\mathbb{P}(x \in X \,|\, \rho, \mathsf{E}) = \langle \mathsf{E}(X), \rho \rangle$ “statistical duality”



  - $\langle \mathscr{V}^*, \mathscr{V} \rangle$ is a dual pair of order unit & base normed Banach spaces
  - $\mathsf{E}(X) \in [0, \mathbb{1}] \subset \mathscr{V}^*$ (unit interval, “effects”), $\rho \in S \subset \mathscr{V}$ (states)

Quantum observables — definition

- An observable extracts classical information from a quantum system.
  - Classical outcomes: $\mathscr{X}$ — a locally compact Hausdorff space
  - Quantum system: $\mathscr{H}$ — a complex separable Hilbert space
- Observable: a weak-* σ-additive measure $\mathsf{E} : \mathscr{F}(\mathscr{X}) \to \mathscr{B}(\mathscr{H})_+$ with $\mathsf{E}(\mathscr{X}) = \mathbb{I}$. ($\mathscr{F}(\mathscr{X})$ = Borel σ-algebra.)

  positive operators
  - If E is measured on a quantum state $\rho \in \mathscr{B}(\mathscr{H})_* = \mathscr{T}(\mathscr{H})$, then

$$\mathbb{P}(x \in X \,|\, \rho, \mathsf{E}) = \langle \rho, \mathsf{E}(X) \rangle = \mathrm{tr}[\rho \mathsf{E}(X)]$$

    is the probability of getting outcome in a set $X \in \mathscr{F}(\mathscr{X})$.

Observables — algebraic “channel” picture

- Let $\mathscr{C}(\mathscr{X}, \mathscr{H}) = \{\Phi : C_b(\mathscr{X}) \to \mathscr{B}(\mathscr{H}) \mid \Phi$ bdd positive linear, $\Phi(1) = \mathbb{I}\}$
- An observable $\mathsf{E} : \mathscr{F}(\mathscr{X}) \to \mathscr{B}_+(\mathscr{H})$ defines a $\Phi_\mathsf{E} \in \mathscr{C}(\mathscr{X}, \mathscr{H})$,

$$\Phi_\mathsf{E}(f) = \int f(x)\mathsf{E}(dx) \quad \text{(weak-* convergent integral).}$$

- **Conversely**, let $\Phi \in \mathscr{C}(\mathscr{X}, \mathscr{H})$ be s.t. $\Phi(\infty) = 0$ where

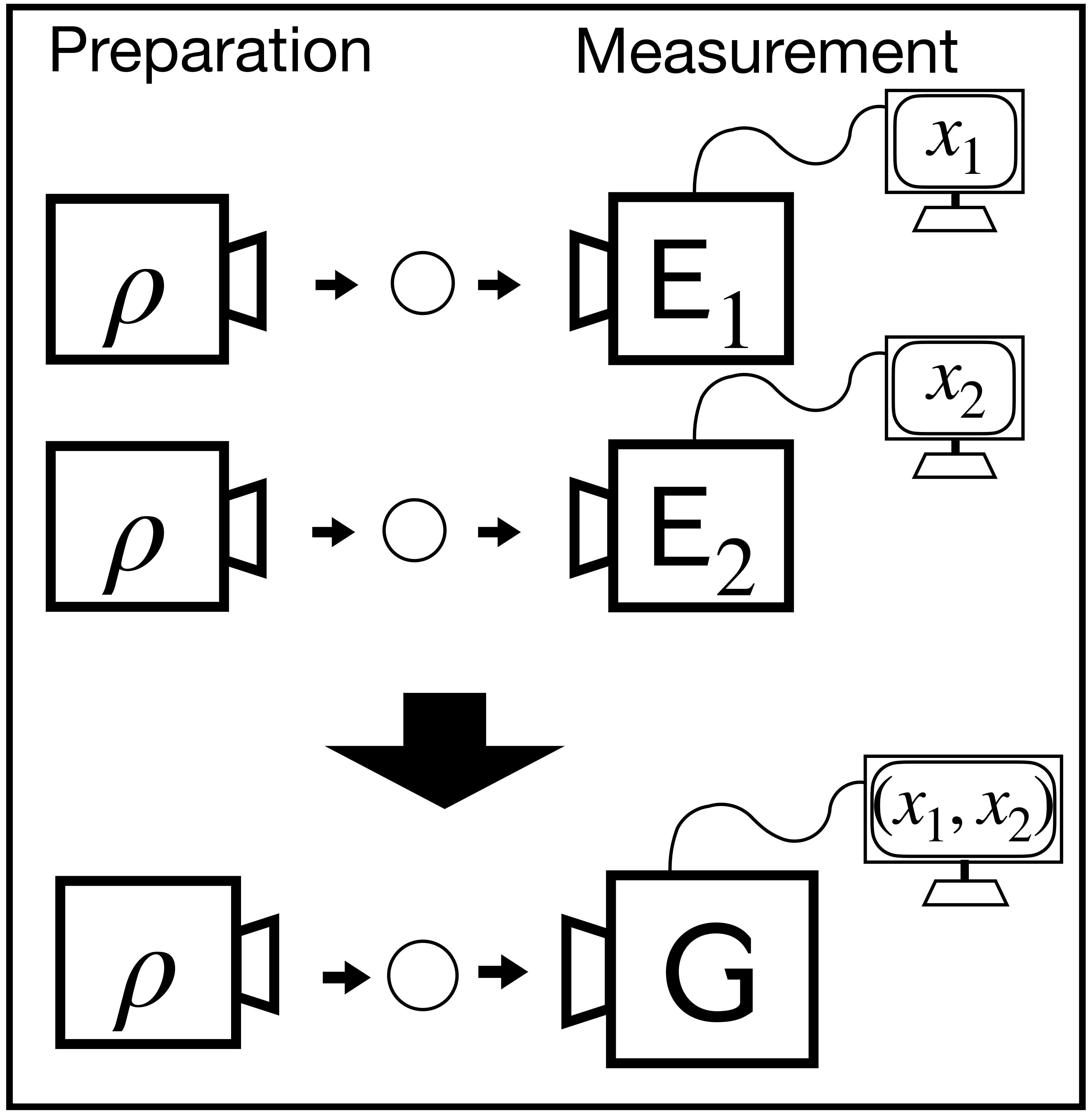
$$\Phi(\infty) = \inf\{\Phi(1-f) \mid f \in C_c(\mathscr{X}), 0 \leq f \leq 1\} \quad \text{(“weight at infinity” [1]).}$$

Then there is a unique observable $\mathsf{E} : \mathscr{F}(\mathscr{X}) \to \mathscr{B}_+(\mathscr{H})$ such that $\Phi = \Phi_\mathsf{E}$.

[1] R. F. Werner, Quant. Inform. Comput. 4, 546–562 (2004)

Joint measurability — conceptual idea

- Can a given pair of observables $\mathsf{E}_1, \mathsf{E}_2$ be simulated by a *single* joint observable G?
- The outcome distribution of G should be a joint probability distribution for the distributions of $\mathsf{E}_1, \mathsf{E}_2$ in every state.
- In quantum theory joint observables do not exist for every pair $\mathsf{E}_1, \mathsf{E}_2$.



Joint measurability — definition

- Definition: Observables $\mathsf{E}_i : \mathscr{F}(\mathscr{X}_i) \to \mathscr{B}_+(\mathscr{H}),\ i = 1,\ldots,J$ are <u>jointly measurable</u> if there is an observable $\mathsf{G} : \mathscr{F}(\mathscr{X}_1 \times \cdots \times \mathscr{X}_J) \to \mathscr{B}_+(\mathscr{H})$ s.t.

$$\Phi_{\mathsf{E}_i} = \Phi_{\mathsf{G}} \circ \Pi_i \text{ for all } i = 1,\ldots,J,$$

where $\Pi_i : C_b(\mathscr{X}_i) \to C_b(\mathscr{X}_1 \times \cdots \times \mathscr{X}_J)$ is the canonical injection.

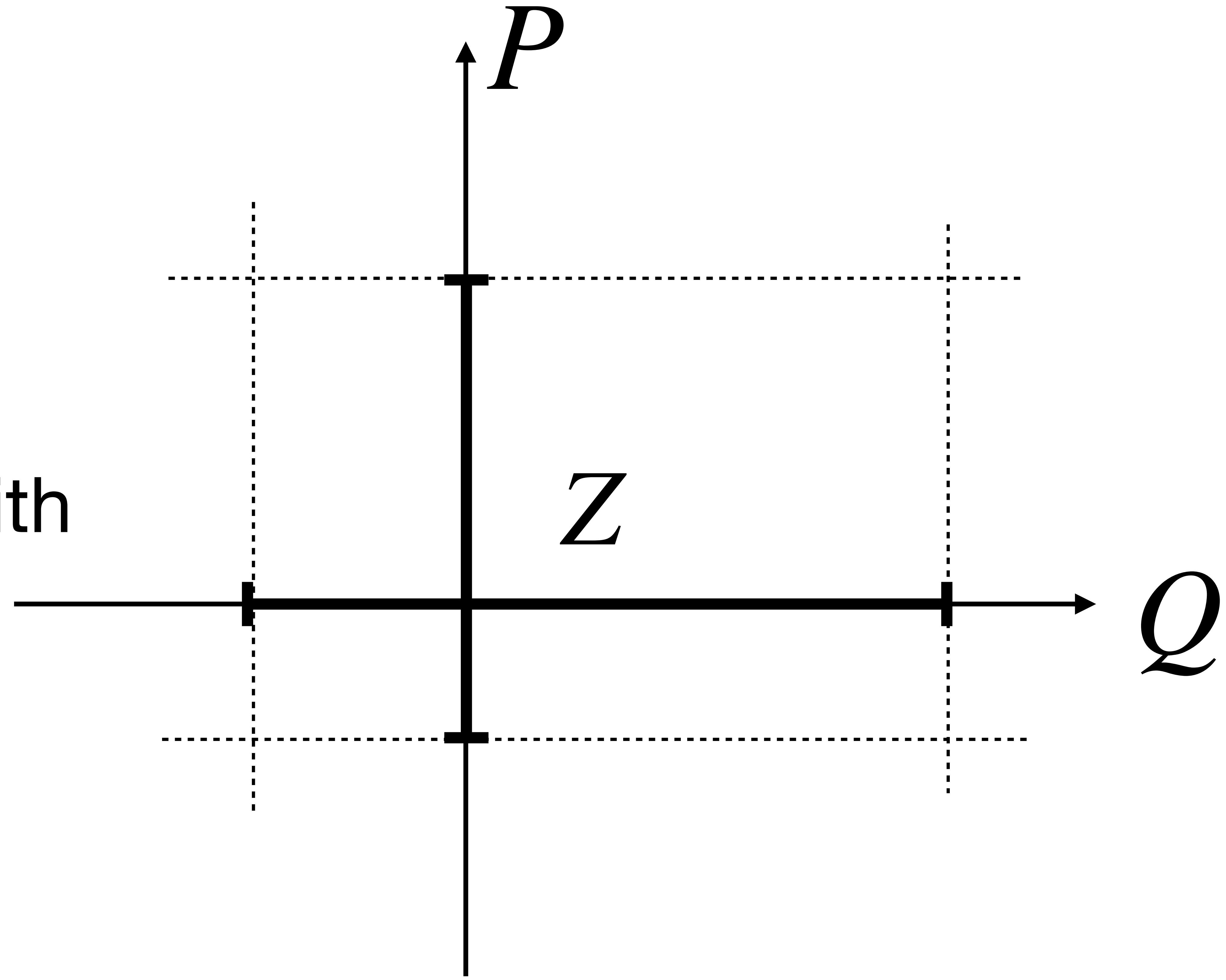
- If E_i are spectral measures, they are jointly measurable iff they commute. A joint observable is $\mathsf{G}(X_1 \times \cdots \times X_J) = \mathsf{E}_1(X_1)\cdots\mathsf{E}_J(X_J) \geq 0,\ X_i \in \mathscr{F}(\mathscr{X}_i)$.
- In general, commutativity is sufficient but **not** necessary for joint measurability.

Joint measurements — phase space setting

- **Basic case: "mixed state localisation" of position-momentum (or time-frequency):**

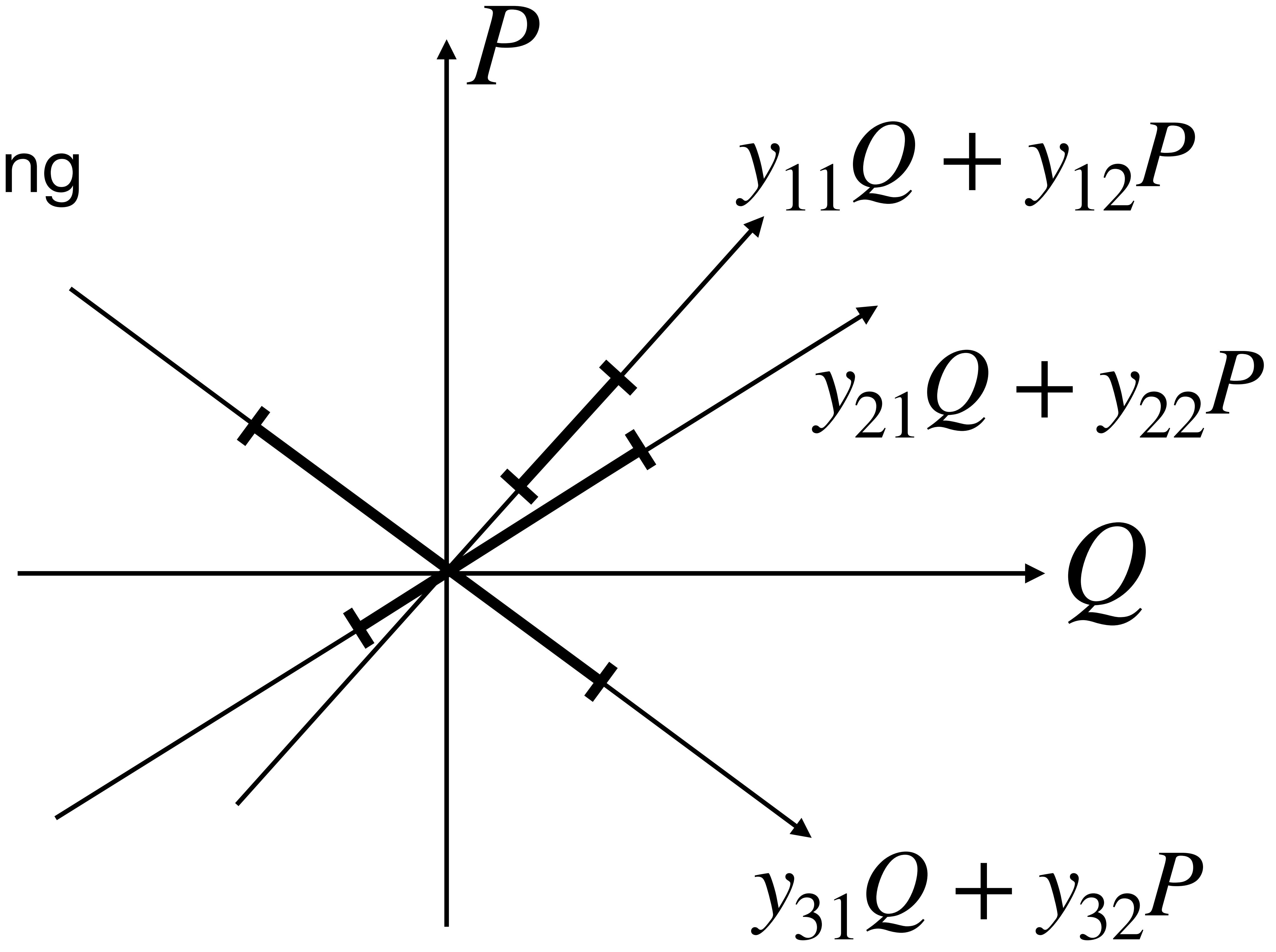
$$\mathsf{G}(Z) = \chi_Z * T$$

  - Integrating over momentum gives a noisy (convolved) position observable $\mu_\rho * \mathsf{Q}$ with $\mu_T(X) = \mathrm{tr}[T\mathsf{P}(-X)]$
  - Similarly, integrating over position gives a noisy momentum $\nu_T * \mathsf{P}$.
  - So the <u>mixed state localisation defines a joint observable</u> for $\mu_T * \mathsf{Q}$ and $\nu_T * \mathsf{P}$.



Joint measurements — phase space setting

- **Generalisation: multiple directions**
  - Look at multiple observables corresponding to arbitrary directions in phase space
  - These are used in physics, say, for state tomography in quantum optics
  - Related to e.g. Radon transform
  - The joint measurability problem now corresponds to localisation in a higher dimensional “hybrid” phase space with degenerate symplectic form



Covariant joint measurability

- Denote $\mathscr{C}(\mathscr{H}) = \{\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H}) \mid \Phi$ (completely) positive linear, $\Phi(\mathbb{1}) = \mathbb{1}\}$, $\mathscr{X}_0 = \mathscr{X}_1 \times \cdots \times \mathscr{X}_J$, $\Pi_0 = \mathrm{Id}_{C_b(\mathscr{X}_0)}$, $\Pi_i : C_b(\mathscr{X}_i) \to C_b(\mathscr{X}_0)$ canonical injections.
- Let an amenable semigroup S act by
  - anti-homomorphisms $s \mapsto \alpha_s^i \in \mathscr{C}(\mathscr{X}_i)$ satisfying $\alpha_s^0 \circ \Pi_i = \Pi_i \circ \alpha_s^i$;
  - a homomorphism $s \mapsto \alpha_s \in \mathscr{C}(\mathscr{H})$ where each α_s is weak-* continuous.
- Call $\Phi \in \mathscr{C}(\mathscr{X}_i, \mathscr{H})$ *covariant* if $\alpha_s \circ \Phi \circ \alpha_s^i = \Phi$ for all $s \in S$.
- **Thm**: *Let $\Phi_0 \in \mathscr{C}(\mathscr{X}_0, \mathscr{H})$. If $\Phi_0 \circ \Pi_i \in \mathscr{C}(\mathscr{X}_i, \mathscr{H})$ is covariant for each $i = 1, \ldots, J$, then there is a covariant $\Phi \in \mathscr{C}(\mathscr{X}_0, \mathscr{H})$ s.t. $\Phi \circ \Pi_i = \Phi_0 \circ \Pi_i$ for each i.* (Proof sketch: use [1] with suitable weak-* topology.)

[1] M. M. Day, "Fixed-point theorems for compact convex sets" Illinois J. Math. 5 585-590, (1961)

Covariant joint measurability

Thm: *If* $\mathsf{E}_i : \mathscr{F}(\mathscr{X}_i) \to \mathscr{B}_+(\mathscr{H}),\ i = 1,\ldots,J$ *are covariant jointly measurable observables, then they have a covariant joint observable* $\mathsf{G} : \mathscr{F}(\mathscr{X}_1 \times \cdots \times \mathscr{X}_J) \to \mathscr{B}_+(\mathscr{H})$.

Proof sketch: Now $\Phi \circ \Pi_i = \Phi_{\mathsf{E}_i}$ for a covariant $\Phi \in \mathscr{C}(\mathscr{X}_0, \mathscr{H})$ [by the preceding Thm]. Additionally $\Phi_{\mathsf{E}_i}(\infty) = 0$ for all i, which implies $\Phi(\infty) = 0$, hence $\Phi = \Phi_{\mathsf{G}}$ for some observable G.

- This result generalises [1] which was based on the ideas from [2,3]

[1] C. Carmeli, T. Heinonen, A. Toigo, J. Phys. A: Math. Gen. 38 5253 (2005)
[2] P. Busch. Internat. J. Theoret. Phys. 24 63–92 (1985)
[3] R. F. Werner, Quant. Inform. Comput. 4, 546–562 (2004).

Phase space

- The phase space is $\Xi = \mathbb{R}^{2N}$.
- Symplectic form $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\intercal \boldsymbol{\Omega} \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \Xi$:

$$\boldsymbol{\Omega} = \bigoplus_{i=1}^{N} \boldsymbol{\Omega}_i, \quad \boldsymbol{\Omega}_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- Hilbert space $\mathscr{H} = L^2(\mathbb{R}^N)$, basic quadratures $\mathbf{R} = (Q_1, P_1, \ldots, Q_N, P_N)^\intercal$ satisfying $[R_i, R_j] = i\Omega_{ij}\,\mathbb{I}$.
- Weyl operators (= time-frequency shifts) $W(\mathbf{x}) := e^{i\mathbf{x}^T\mathbf{R}}$ with CCR

$$W(\mathbf{x})W(\mathbf{y}) = e^{-i\mathbf{x}^T\boldsymbol{\Omega}\mathbf{y}}W(\mathbf{y})W(\mathbf{x}).$$

Quasi-free observables

- Translations: for any $\mathbf{x} \in \Xi$ define

$$\alpha_{\mathbf{x}}(A) = W(\boldsymbol{\Omega}\mathbf{x})^* A W(\boldsymbol{\Omega}\mathbf{x}) \text{ for } A \in \mathscr{B}(\mathscr{H}) \quad \text{[quantum]}$$

$$\alpha_{\mathbf{x}}(f)(\mathbf{r}) = f(\mathbf{r} + \mathbf{x}) \text{ for } f \in C_b(\mathbb{R}^m) \quad \text{[classical]}$$

- Definition [1]: Let $\mathbf{S} : \mathbb{R}^m \to \Xi$ be any linear map. An observable $\mathsf{E} : \mathscr{F}(\mathbb{R}^m) \to \mathscr{B}_+(\mathscr{H})$ is $\mathbf{S}$-*covariant*, if

$$\Phi_{\mathsf{E}} \circ \alpha_{\mathbf{S}^T\mathbf{x}} = \alpha_{\mathbf{x}} \circ \Phi_{\mathsf{E}} \quad \text{for all } \mathbf{x} \in \Xi.$$

  An observable is *quasi-free* if it is $\mathbf{S}$-covariant for some $\mathbf{S}$.

- Example: mixed state localisation $\mathsf{G}(Z) = \chi_Z * T$ is $\mathbf{S}$-covariant for $\mathbf{S} = \mathbb{I}$

[1] R. F. Werner, L. Dammeier, Quantum 7, 1068 (2023).

Structure of quasi-free observables

- Definition [1]: an observable $\mathsf{E} : \mathscr{F}(\mathbb{R}^m) \to \mathscr{B}_+(\mathscr{H})$ is $\mathbf{S}$-*covariant*, if

$$\Phi_{\mathsf{E}} \circ \alpha_{\mathbf{S}^T\mathbf{x}} = \alpha_{\mathbf{x}} \circ \Phi_{\mathsf{E}} \quad \text{for all } \mathbf{x} \in \Xi$$

- Theorem [1]: Any $\mathbf{S}$-covariant observable is determined, through

$$\Phi_{\mathsf{E}}(e^{i\mathbf{t}^\intercal(\cdot)}) = h(\mathbf{t})W(\mathbf{S}\mathbf{t}),$$

by some function $h : \mathbb{R}^m \to \mathbb{C}$ with the "twisted definite" property:

For any $(\mathbf{x}_i)_{i=1}^k \subset \mathbb{R}^m$ the matrix

$$H_{ij} := h(-\mathbf{x}_i + \mathbf{x}_j)e^{-i\frac{1}{2}\mathbf{x}_i^T\tilde{\boldsymbol{\Omega}}\mathbf{x}_j}$$

is positive semidefinite, where $\tilde{\boldsymbol{\Omega}} = -\mathbf{S}^T\boldsymbol{\Omega}\mathbf{S}$. ← a possibly degenerate symplectic form

[1] R. F. Werner, L. Dammeier, Quantum 7, 1068 (2023).

Structure of quasi-free observables

- If $\ker \mathbf{S} = \{0\}$, "quantum Bochner's theorem" [1] gives a mixed state T s.t. $h(\mathbf{x}) = \hat{T}(\mathbf{x}) := \mathrm{tr}[\tilde{W}(\mathbf{x})T]$, where $\tilde{W}$ is the Weyl rep for the phase space $(\mathbb{R}^m, \tilde{\mathbf{\Omega}})$ with $\tilde{\mathbf{\Omega}} = -\mathbf{S}^T\mathbf{\Omega}\mathbf{S}$.

- If $\ker \mathbf{S} \neq \{0\}$, the phase space $(\mathbb{R}^m, \tilde{\mathbf{\Omega}})$ is a *hybrid* with commutative degrees of freedom: $\mathbb{R}^m = \Xi_q \oplus \Xi_c$ where $\Xi_c = \ker \tilde{\mathbf{\Omega}} = \ker \mathbf{S}$.

- Bochner's theorem for hybrids [2]: $\exists$ *hybrid state* T such that

$$h(\mathbf{x}) = \hat{\rho}(\mathbf{x}) = \hat{\rho}(\mathbf{x}_q \oplus \mathbf{x}_c) = \int_{\Xi_c} d\mu(\mathbf{r})\, e^{i\mathbf{x}_c^T\mathbf{r}}\, \mathrm{tr}[T_{\mathbf{r}}\tilde{W}_q(\mathbf{x}_q)]$$

Hybrid state = measure & family of density operators

- Quasi-free observables correspond to pairs $(\mathbf{S}, T)$.

[1] R. Werner, J. Math. Phys. 25 1404 (1984)

[2] R. F. Werner, L. Dammeier, Quantum 7, 1068 (2023).

Quasi-free joint measurability

- For each $i = 1,\ldots,J$ let $\mathsf{E}_i : \mathscr{F}(\mathbb{R}^{m_i}) \to \mathscr{B}_+(\mathscr{H})$ be quasi-free with $(\mathbf{S}_i, T_i)$.
- Joint outcome set $\mathbb{R}^m \simeq \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_J}$ with projections $\mathbf{P}_i : \mathbb{R}^m \to \mathbb{R}^{m_i}$.
- Consider an observable $\mathsf{G} : \mathscr{F}(\mathbb{R}^m) \to \mathscr{B}_+(\mathscr{H})$ quasi-free with $(\mathbf{S}, T)$. Then

G is a joint observable for the E_i

$\Leftrightarrow\ \Phi_{\mathsf{G}}(f \circ \mathbf{P}_i) = \Phi_{\mathsf{E}_i}(f)$ for all $i = 1,\ldots,J,\ f \in C_b(\mathbb{R}^{m_i})$

$\Leftrightarrow\ \Phi_{\mathsf{G}}(e^{i\mathbf{t}^T\mathbf{P}_i(\cdot)}) = \Phi_{\mathsf{E}_i}(e^{i\mathbf{t}^T(\cdot)})$ for all $i = 1,\ldots,J,\ \mathbf{t} \in \mathbb{R}^{m_i}$

$\Leftrightarrow\ \hat{T}(\mathbf{P}_i^T\mathbf{t})W(\mathbf{S}\mathbf{P}_i^T\mathbf{t}) = \hat{T}_i(\mathbf{t})W(\mathbf{S}_i\mathbf{t})$ for all $i = 1,\ldots,J,\ \mathbf{t} \in \mathbb{R}^{m_i}$

$\Leftrightarrow\ \mathbf{S} = \begin{pmatrix}\mathbf{S}_1 & \cdots & \mathbf{S}_J\end{pmatrix}$ and $\hat{T} \circ \mathbf{P}_i^T = \hat{T}_i$ for all $i = 1,\ldots,J$.

Quasi-free joint measurability

- For each $i = 1, \ldots, J$ let $\mathsf{E}_i : \mathscr{F}(\mathbb{R}^{m_i}) \to \mathscr{B}_+(\mathscr{H})$ be quasi-free with $(\mathbf{S}_i, T_i)$. Then:

  The E_i are jointly measurable

  $\Leftrightarrow$ The E_i have a quasi-free joint observable [by covariance]

  $\Leftrightarrow$ $\exists$ hybrid state T s.t. $\hat{T}_i = \hat{T} \circ \mathbf{P}_i^T$ for all $i = 1, \ldots, J$.

- In this case the pair $(\mathbf{S}, T)$ gives a joint observable.

Quasi-free joint measurability is a marginal problem for hybrid states.

Joint measurements are generalisations of mixed state localisation.

Joint measurability of isotropic localisations

- An *isotropic localisation* is an $\mathbf{S}$-covariant observable with $\tilde{\mathbf{\Omega}} = -\mathbf{S}^\intercal \mathbf{\Omega} \mathbf{S} = \mathbf{0}$.
- A quasi-free observable is isotropic iff its noise state is classical (i.e. a measure).
- $\mathbf{t} \mapsto W(\mathbf{S}\mathbf{t})$ is a unitary group when $\mathbf{S}^\intercal \mathbf{\Omega} \mathbf{S} = \mathbf{0}$ (by CCR). The spectral measure $\mathsf{Q}_{\mathbf{S}} : \mathscr{F}(\mathbb{R}^m) \to \mathscr{B}_+(\mathscr{H})$ of its Stone generator is a "noiseless" isotropic localisation.
- All other cases obtained by convolution with a probability measure:

$$(\mu * \mathsf{Q}_{\mathbf{S}})(X) := \int \mu(X - \mathbf{r})\, \mathsf{Q}_{\mathbf{S}}(d\mathbf{r}); \qquad \widehat{(\mu * \mathsf{Q}_{\mathbf{S}})}(\mathbf{t}) = \hat{\mu}(\mathbf{t})\,\widehat{\mathsf{Q}_{\mathbf{S}}}(\mathbf{t}) = \hat{\mu}(\mathbf{t}) W(\mathbf{S}\mathbf{t})$$

  - $\mu * \mathsf{Q}_{\mathbf{S}}$ is an isotropic localisation with noise state μ.
- **Question**: Take J noiseless isotropic localisations with matrices $\mathbf{S}_i : \mathbb{R}^{m_i} \to \Xi$. For which measures μ_i are their noisy versions $\mu_1 * \mathsf{Q}_{\mathbf{S}_1}, \ldots, \mu_J * \mathsf{Q}_{\mathbf{S}_J}$ jointly measurable?

Joint measurability of isotropic localisations

- **Thm**: *Let* $\mathbf{S}_i : \mathbb{R}^{m_i} \to \Xi$ *be matrices with* $\mathbf{S}_i^\intercal \boldsymbol{\Omega} \mathbf{S}_i = \mathbf{0}$ *and* μ_i *noise measures. Denote* $\mathbf{S} = \begin{pmatrix} \mathbf{S}_1 & \cdots & \mathbf{S}_J \end{pmatrix}$, *assume* $\operatorname{rank} \mathbf{S} = 2N$. *The following are equivalent:*

  (i) $\mu_1 * \mathsf{Q}_{\mathbf{S}_1}, \ \ldots, \mu_n * \mathsf{Q}_{\mathbf{S}_n}$ *are jointly measurable;*

  (ii) $\mu_i = \operatorname{tr}[S_i \mathsf{Q}_{-\mathbf{S}_i}(\,\cdot\,)]$ *where* $S_i = \int_{\ker \mathbf{S}} \nu(d\mathbf{r}) \ \alpha_{\mathbf{V}_i \mathbf{P}_i \mathbf{r}}(T_\mathbf{r})$ *for a positive measure* ν *and an integrable positive trace-class valued function* $\mathbf{r} \mapsto T_\mathbf{r}$ *on* $\ker \mathbf{S}$.

Here $\mathbf{V}_i : \mathbb{R}^{m_i} \to \Xi$ is the matrix with $\operatorname{ran} \mathbf{V}_i = \operatorname{ran} \mathbf{S}_i$ and $\mathbf{S}_i^\intercal \mathbf{V}_i = \mathbf{I}_{m_i}$.

Necessary condition for quadratures

- Take $\mathbf{y}_i \in \Xi = \mathbb{R}^2$ such that $\mathrm{span}\{\mathbf{y}_1, \ldots, \mathbf{y}_J\} = \Xi$. Consider quadratures $Q_{\mathbf{y}_i} = \mathbf{y}_i^\intercal \mathbf{R} = y_{i1}Q + y_{i2}P$, and let μ_i be probability measures on $\mathbb{R}$.

- **Thm:** *If* $\mu_1 * \mathsf{Q}_{\mathbf{y}_1}, \ \ldots, \mu_J * \mathsf{Q}_{\mathbf{y}_J}$ *are jointly measurable, the noise measures satisfy the uncertainty relation*

$$\sum_{i=1}^{J} \mathrm{Var}(\mu_i) \geq \frac{1}{\sqrt{2}} \|\tilde{\mathbf{\Omega}}\|_2, \qquad \textit{where } \tilde{\mathbf{\Omega}}_{ij} = -\mathbf{y}_i^\intercal \mathbf{\Omega} \mathbf{y}_j.$$

Proof: Follows from the general result combined with an UR for multiple quadratures from [1].

  - Joint measurability requires (at least) certain amount of noise.

[1] S. Kechrimparis, S. Weigert, J. Phys. A: Math. Theor. 51 025303 (2018)

Thank you