Quantum harmonic analysis on spaces of analytic functions
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Joint work with

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Spaces of analytic functions

- Let $\Omega \subset \mathbb{C}^n$ be a domain and let μ be a probability measure on Ω. Then $\mathcal{A}^2(\Omega, \mu) = \mathcal{O}(\Omega) \cap L^2(\Omega, \mu)$ is a RKHS. For $f \in \mathcal{A}^2(\Omega, \mu)$,

$$f(z) = \langle f, K_z \rangle, \quad z \in \Omega.$$

- The Bergman projection $P : L^2(\Omega, \mu) \to \mathcal{A}^2(\Omega, \mu)$ is given by

$$Pf(z) = \langle f, K_z \rangle.$$

- Examples:
  1. The Bergman space $\mathcal{A}^2(\mathbb{B}^n)$

$$K(w, z) = K_z(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1}} \quad w, z \in \mathbb{B}^n,$$

  2. The Fock space $\mathcal{F}^2(\mathbb{C}^n)$ with $d\mu(z) = e^{-\pi|z|^2} dz$, $K_z(w) = e^{\pi \langle w, z \rangle}$.

Toeplitz algebra

- For $a : \Omega \to \mathbb{C}$, the Toeplitz operator $T_a : \mathcal{A}^2(\Omega, \mu) \to \mathcal{A}^2(\Omega, \mu)$ is

$$T_a f(z) = P(af)(z) = \langle af, K_z \rangle$$

If $a \in L^\infty(\mathbb{B}^n)$, T_a is a bounded operator with $\|T_a\| \leq \|a\|_\infty$.

- The Toeplitz algebra $\mathfrak{T}(L^\infty)$ is the C^*-algebra generated by T_a with $a \in L^\infty(\Omega)$.
- (Xia 2015) Toeplitz operartors are dense in $\mathfrak{T}(L^\infty)$ for both $\mathcal{A}^2(\mathbb{B}^n)$ and $\mathcal{F}^2(\mathbb{C}^n)$.
  $\Longrightarrow$ Q: How to approximate $S \in \mathfrak{T}(L^\infty)$ by Toeplitz operators

Toeplitz algebra

- For $a : \Omega \to \mathbb{C}$, the Toeplitz operator $T_a : \mathcal{A}^2(\Omega, \mu) \to \mathcal{A}^2(\Omega, \mu)$ is

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- (Xia 2015) Toeplitz operartors are dense in $\mathfrak{T}(L^\infty)$ for both $\mathcal{A}^2(\mathbb{B}^n)$ and $\mathcal{F}^2(\mathbb{C}^n)$.
  $\implies$ Q: How to approximate $S \in \mathfrak{T}(L^\infty)$ by Toeplitz operators
- If $S \in \mathcal{L}(\mathcal{A}^2)$, the Berezin transform of S is given by:

$$B(S)(z) = \langle Sk_z, k_z \rangle, \ z \in \Omega$$

where $k_z = \frac{K_z}{\|K_z\|}$.

QHA setup

- A locally compact unimodular group G acts on Ω.
- An irreducible square-integrable projective unitary representation of G of the form:

$$(\pi(g)f)(z) = j(g^{-1}, z)f(g^{-1}z), \quad \forall z \in \Omega, f \in \mathcal{A}^2(\Omega)$$

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  where j is a cocycle.
- Examples:
  - $\mathcal{F}^2(\mathbb{C}^n)$, $G = \mathbb{C}^n$, Weyl representation
  - $\mathcal{A}^2(\mathbb{B}^n)$, $G = SU(n,1)$ (matrices in $SL(n+1,\mathbb{C})$ that preserves the sesquilinear form $\langle z, w\rangle_{n,1} := -z_1\bar{w}_1 - \cdots - z_n\bar{w}_n + z_{n+1}\bar{w}_{n+1}$)
    $SU(n,1)$ acts on $\mathbb{B}^n$ by the fractional linear transformations given by

$$\begin{bmatrix} A & v \\ w^t & c \end{bmatrix} \cdot z = \frac{Az + v}{w^t z + c}, \quad z \in \mathbb{B}^n.$$

    Then $\mathbb{B}^n = G/K$, where $K = U_n$ and π is the discrete series representation.

Convolutions of functions on $\mathbb{B}^n$

- If $\psi : G \to \mathbb{C}$ and $a : \mathbb{B}^n \to \mathbb{C}$, the convolution $\psi * a : \mathbb{B}^n \to \mathbb{C}$ is defined formally by

$$(\psi * a)(z) := \int_G \psi(g) a(g^{-1}z) \, d\mu_G(g), \quad \forall z \in \mathbb{B}^n,$$

  $d\mu_G$ is the Haar measure on G.
- The convolution is noncommutative. But if both a and ψ are radial functions on $\mathbb{B}^n$, $\psi * a = a * \psi$ (a is radial if $a(k^{-1}z) = a(z)$ for all $k \in U(n)$, $z \in \mathbb{B}^n$), i.e. G/K is a commutative space.

Convolution of a function and an operator

- Translations of operators: For $S \in \mathcal{L}(\mathcal{A}^2)$, translation of S by $g \in G$ is given by

$$L_g(S) = \pi(g) S \pi(g)^*.$$

  An operator S is radial if $L_k(S) = S$ for all $k \in U(n)$.

- For $\psi : G \to \mathbb{C}$ and $S \in \mathcal{L}(\mathcal{A}^2)$, define the convolution $\psi * S$ in weak sense by

$$\psi * S := \int_G \psi(g) L_g(S) \, d\mu_G(g)$$

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- Toeplitz operators are convolutions: Let $\Phi = 1 \otimes \bar{1}$.

$$T_a = a * \Phi.$$

Convolution between two opertors

- Let $S \in \mathcal{L}(\mathcal{A}^2)$ and A be trace class. Then $S * A : G \to \mathbb{C}$ is given by

$$(S * A)(g) := \mathrm{Tr}(SL_g(A)) \quad \forall g \in G.$$

  Then $S * A \in L^\infty(G)$ and $\|S * A\|_\infty \leq \|S\|\|A\|_1$.
- If $S \in \mathcal{L}(\mathcal{A}^2)$ and A, B are radial trace-class class operators

$$(S * A) * B = (S * B) * A.$$

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- If $S \in \mathcal{L}(\mathcal{A}^2)$ and A, B are radial trace-class class operators
$$(S * A) * B = (S * B) * A.$$
- For $S \in \mathcal{L}(\mathcal{A}^2)$, the Berezin transform of S is given by
$$B(S) = S * \Phi.$$

Part I: Approximations by Toeplitz operators on $\mathcal{A}^2(\mathbb{B}^n)$

(Joint work with Matthew Dawson, Mishko Mitkovski and Gestur Ólafsson)

- Introduce a new α-Berezin transform
- Characterize the radial Toeplitz algebra
- Discuss a Wiener's Tauberian theorem.
- Approximate Schatten-p operators by Toeplitz operators

Suarez's α-Berezin transform for the unit disc

- The α-Berezin transform: For $\alpha \in \mathbb{N}_0$,

$$B_\alpha(S)(z) = C_\alpha(1-|z|^2)^2 \sum_{m=0}^{\alpha}(-1)^m \binom{\alpha}{m} \langle S(p_m k_z^\alpha), p_m k_z^\alpha \rangle, \quad z \in \mathbb{D}.$$

  where $p_m(z) = z^m$ and $k_z^\alpha(w) = \frac{(1-|z|^2)^{(n+1+\alpha)/2}}{(1-\langle w,z\rangle)^{n+1+\alpha}}$.
- $B_\alpha(T_a) = B_\alpha(a) = \langle a k_z^\alpha, k_z^\alpha \rangle_\alpha$.

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  where $p_m(z) = z^m$ and $k_z^\alpha(w) = \frac{(1-|z|^2)^{(n+1+\alpha)/2}}{(1-\langle w,z\rangle)^{n+1+\alpha}}$.

- $B_\alpha(T_a) = B_\alpha(a) = \langle a k_z^\alpha, k_z^\alpha \rangle_\alpha$.

Conjecture: If $S \in \mathfrak{T}(L^\infty)$ then $T_{B_\alpha(S)} \to S$ in operator norm as $\alpha \to \infty$.

- (Suarez 2005) A radial operator S is in the Toeplitz algebra iff $T_{B_\alpha(S)} \to S$ in operator norm (unit disc).
- (Suarez 2004, 2007) $T_{B_\alpha(a)} \to T_a$ in operator norm.

A natural approximate identity for $L^1(\mathbb{B}^n, d\lambda)$

- The invariant measure λ on $\mathbb{B}^n$ is given by $d\lambda(z) = \frac{1}{(1-|z|^2)^{n+1}} dz$.
- By the identification of functions on $\mathbb{B}^n$ as functions on G, we get that

$$L^1(\mathbb{B}^n, d\lambda) \subset L^1(G).$$

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- We have

$$(\Phi * \Phi)(z) = C_0(1 - |z|^2)^{n+1} =: \varphi(z) \quad \forall z \in \mathbb{B}^n.$$

- The functions φ_α given below, is a right-approximate identity in $L^1(\mathbb{B}^n, d\lambda)$:

$$\varphi_\alpha(z) = C_\alpha(1 - |z|^2)^{n+1+\alpha}, \quad \forall z \in \mathbb{B}^n.$$

- We have $B_\alpha(a) = \langle a k_z^\alpha, k_z^\alpha \rangle = a * \varphi_\alpha$.

A new α-Berezin transform

- We define an operator Φ_α s.t.

$$\Phi_\alpha * \Phi = \varphi_\alpha.$$

  Then Φ_α is a radial finite rank operator. Then $\mathrm{Tr}(\Phi_\alpha) = 1$ but $\|\Phi_\alpha\|_1$ depends on α.
- For $\alpha \in \mathbb{N}_0$ and $S \in \mathcal{L}(\mathcal{A}^2)$, we define

$$\tilde{B}_\alpha(S) = S * \Phi_\alpha.$$

  Then $\tilde{B}_\alpha(S) \in L^\infty(\mathbb{B}^n)$ and $\|\tilde{B}_\alpha(S)\|_\infty \leq \|S\| \|\Phi_\alpha\|_1$.

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- For $\alpha \in \mathbb{N}_0$ and $S \in \mathcal{L}(\mathcal{A}^2)$, we define
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Then $\tilde{B}_\alpha(S) \in L^\infty(\mathbb{B}^n)$ and $\|\tilde{B}_\alpha(S)\|_\infty \leq \|S\| \|\Phi_\alpha\|_1$.
- $\tilde{B}_\alpha(T_a) = B_\alpha(a)$.
- $B_\beta(\tilde{B}_\alpha(S)) = B_\alpha(\tilde{B}_\beta(S))$.
- $S \in \mathcal{L}(\mathcal{A}^2)$ is a Toeplitz operator iff there is $C > 0$ s.t. $\|\tilde{B}_\alpha(S)\|_\infty \leq C$ for all $\alpha \in \mathbb{N}_0$.

Q: Is $T_{\tilde{B}_\alpha(S)} \to S$ in operator norm as $\alpha \to \infty$ for $S \in \mathfrak{T}(L^\infty)$.

Uniform continuity

- $a \in L^\infty(\mathbb{B}^n)$ is **left-G-uniformly continuous** if the map $G \to L^\infty(\mathbb{B}^n)$, $g \mapsto \ell_g a$, is continuous w.r.t. $\|\cdot\|_\infty$.
- a on $\mathbb{B}^n$ is **right-G-uniformly continuous** if the map $G \to L^\infty(G)$, $g \mapsto r_g a$ is continuous w.r.t. $\|\cdot\|_\infty$, where

$$(r_g a)(h) = a(hg \cdot 0), \quad h \in G.$$

- An operator $S \in \mathcal{L}(\mathcal{A}^2)$ is **left-G-uniformly continuous** if the map $G \to \mathcal{L}(\mathcal{A}^2)$, $g \mapsto L_g(S)$ is continuous w.r.t. operator norm.

Let $C_{b,u}^{(L)}(\mathbb{B}^n)$, $C_{b,u}^{(R)}(\mathbb{B}^n)$ and $C_{b,u}^{(L)}(\mathcal{A}^2)$ denote the C* algebras of left and right uniformly continuous functions, and uniformly continuous operators.

Radial operators

Proposition (DDMÓ)

Let $S \in \mathcal{L}(\mathcal{A}^2)$ be a radial operator. Then $\varphi_\alpha * S = T_{\tilde{B}_\alpha(S)}$ and

1. $\varphi_\alpha * S \to S$ *in strong operator topology*
2. $\varphi_\alpha * S \to S$ *in* $\|\cdot\|$ *if* $S \in C_{b,u}^{(L)}(\mathcal{A}^2)$.
3. $\varphi_\alpha * S \to S$ *in* $\|\cdot\|_p$ *if* $S \in \mathcal{S}^p(\mathcal{A}^2)$.

Theorem (DDMÓ)

Radial Toeplitz algebra $\mathfrak{T}(L^\infty)^{Rad}$ can be characterized as

$$\begin{aligned}\mathfrak{T}(L^\infty)^{Rad} &= \{\text{The algebra of all bounded uniformly continuous radial operators}\}\\ &= \{\text{radial } S \in \mathcal{L}(\mathcal{A}^2) \mid T_{\tilde{B}_\alpha(S)} \to S \text{ in operator norm.}\}\end{aligned}$$

Approximations

- $T_{B_\alpha(a)} \to T_a$ in operator norm, because

$$T_{B_\alpha(a)} = (a * \varphi_\alpha) * \Phi = a * (\varphi_\alpha * \Phi) \to a * \Phi.$$

- Toeplitz algebra $\mathfrak{T}(L^\infty)$ is generated by Toeplitz operators with bounded right-uniformly continuous symbols.

Q: $\mathfrak{T}(L^\infty)=$ "some algebra of right continuous operators"?.

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Q: $\mathfrak{T}(L^\infty)$= "some algebra of right continuous operators"?.

Proposition

Let $1 \leq p < \infty$. Then we have the following:

1. *If $S \in L^p(\mathbb{B}^n, d\lambda) * \mathcal{S}^1(\mathcal{A}^2)$ then $T_{\tilde{B}_\alpha(S)} \to S$ in Schatten-p norm.*
2. *If $S \in L^1(\mathbb{B}^n, d\lambda) * \mathcal{S}^p(\mathcal{A}^2)$ then $T_{\tilde{B}_\alpha(S)} \to S$ in Schatten-p norm.*
3. *If $S \in L^1(\mathbb{B}^n, d\lambda) * C_{b,u}^{(L)}(\mathcal{A}^2)$ then $T_{\tilde{B}_\alpha(S)} \to S$ in operator norm.*
4. *If $S \in L^\infty(\mathbb{B}^n) * \mathcal{S}^1(\mathcal{A}^2)$ then $T_{\tilde{B}_\alpha(S)} \to S$ in operator norm.*

QHA Wiener's Tauberian theorem

- A function $\psi \in L^p(\mathbb{B}^n, d\lambda)$ is *p*-cyclic (*p*-regular) if the translates of ψ span a dense subset of $L^p(\mathbb{B}^n, d\lambda)$.
- An operator $S \in \mathcal{S}^p(\mathcal{A}^2)$ is *p*-cyclic if translates of S spans a dense subset of $\mathcal{S}^p(\mathcal{A}^2)$.

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- A function $\psi \in L^p(\mathbb{B}^n, d\lambda)$ is p-cyclic (p-regular) if the translates of ψ span a dense subset of $L^p(\mathbb{B}^n, d\lambda)$.
- An operator $S \in \mathcal{S}^p(\mathcal{A}^2)$ is p-cyclic if translates of S spans a dense subset of $\mathcal{S}^p(\mathcal{A}^2)$.

Theorem (Wiener's Tauberian theorem for $L^p(G/K) = L^p(\mathbb{B}^n, d\lambda)$)

Let $1 \leq p < \infty$ and let $\Psi \in \mathcal{S}^p(\mathcal{A}^2)$ be a radial operator. Then the following are equivalent:

1. Ψ *is p-cyclic*
2. $\mathcal{S}^p(\mathcal{A}^2) = \overline{L^1(\mathbb{B}^n, d\lambda) * \Psi}^{\mathcal{S}^p}$
3. $S \mapsto S * \Psi$ *is injective from* $\mathcal{S}^q(\mathcal{A}^2) \to L^\infty(\mathbb{B}^n)$.
4. $a \mapsto a * \Psi$ *is injective from* $L^q(\mathbb{B}^n, d\lambda) \to \mathcal{L}(\mathcal{A}^2)$
5. $L^p(\mathbb{B}^n, d\lambda) = \overline{\mathcal{S}^1(\mathcal{A}^2) * \Psi}^{L^p}$.

Following is similar to (Luef,Skrettingland 18)

Theorem (Wiener's Tauberian theorem- part II)

Let $1 \leq p < \infty$ *and let* $\Psi \in \mathcal{S}^1(\mathcal{A}^2)$ *be radial then the following are equivalent:*

1. Ψ *is p-cyclic*
2. $\mathcal{S}^p(\mathcal{A}^2) = \overline{L^p(\mathbb{B}^n, d\lambda) * \Psi}^{\mathcal{S}^p}$
3. $S \mapsto S * \Psi$ *is injective from* $\mathcal{S}^q(\mathcal{A}^2) \to L^q(\mathbb{B}^n, d\lambda)$.
4. $a \mapsto a * \Psi$ *is injective from* $L^q(\mathbb{B}^n, d\lambda) \to \mathcal{S}^q(\mathcal{A}^2)$
5. $L^p(\mathbb{B}^n, d\lambda) = \overline{\mathcal{S}^p(\mathcal{A}^2) * \Psi}^{L^p}$.

Consequences

Corollary

Let $1 \leq p < \infty$. *Then*

1. $\mathcal{S}^p(\mathcal{A}^2) = \overline{L^p(\mathbb{B}^n, d\lambda) * \Phi}^{\mathcal{S}^p} = \overline{\{T_\psi \mid \psi \in L^p(\mathbb{B}^n, d\lambda)\}}^{\mathcal{S}^p}$
2. $L^p(\mathbb{B}^n, d\lambda) = \overline{\mathcal{S}^p(\mathcal{A}^2) * \Phi}^{L^p} = \overline{\{B(S) \mid S \in \mathcal{S}^p(\mathcal{A}^2)\}}^{L^p}$.

Consequences

Corollary

Let $1 \le p < \infty$. *Then*

1. $\mathcal{S}^p(\mathcal{A}^2) = \overline{L^p(\mathbb{B}^n, d\lambda) * \Phi}^{\mathcal{S}^p} = \overline{\{T_\psi \mid \psi \in L^p(\mathbb{B}^n, d\lambda)\}}^{\mathcal{S}^p}$
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Theorem (DDMÓ)

If $S \in \mathcal{S}^p(\mathcal{A}^2)$ *then* $T_{\tilde{B}_\alpha(S)} \to S$ *in Schatten-p norm.*

Part II: Gelfand theory of radial Toeplitz algebra on $\mathcal{F}^2(\mathbb{C}^n)$.

(Joint work with Mishko Mitkovski)

- Domain of the Laplacian is dense in $\mathfrak{T}(L^\infty)$.
- Revisit Gelfand theory of $\mathfrak{T}(L^\infty)^{U(n)}$
  - (Grudsky, Vasilevski 2002): $\mathfrak{T}(L^\infty)^{U(n)}$ is commutative and the eigenvalue sequences of radial Toeplitz operators on $\mathcal{F}^2(\mathbb{C})$ are of the form

$$\gamma_a(m) = \frac{1}{m!}\int_0^\infty a(\sqrt{r})r^m e^{-r}\,dr.$$

  - (Esmeral, Maximenko 2016): Radial Toeplitz operators are dense in $\mathfrak{T}(L^\infty)^{U(n)}$ and $\mathfrak{T}(L^\infty)^{U(n)}$ is isometrically isomorphic to the C^*-algebra $\mathcal{C}_{b,u}(\mathbb{N}_0, \rho)$ of bounded sequences uniformly continuous w.r.t. the square-root metric $\rho : \mathbb{N}_0 \times \mathbb{N}_0 \to [0, \infty)$ given by

$$\rho(m, m') = |\sqrt{m} - \sqrt{m'}|.$$

Operator Laplacian

- Define the *domain of the Laplacian D_Δ* by

$$D_\Delta := \{S \in \mathcal{L}(\mathcal{A}^2) \mid \exists T \in \mathcal{L}(\mathcal{A}^2) \text{ s.t. } \Delta B(S) = B(T)\}.$$

  Define the *Laplacian of operators* $\Delta : D_\Delta \to \mathcal{L}(\mathcal{A}^2)$, by

$$\Delta S := T$$

  for $S \in D_\Delta$, where T is the operator that satisfies $\Delta B(S) = B(T)$. (Suarez 08).
- $T_a \in D_\Delta$.

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- $T_a \in D_\Delta$.

Theorem (D,Mitkovski 24)

$$\mathfrak{T}(L^\infty) = \overline{D_\Delta}$$

Proof

- Let $\varphi_t(z) = \frac{1}{(\pi t)^n} e^{-\frac{1}{t}|z|^2}$. Then for $a \in L^\infty(\mathbb{B}^n)$,

$$\Delta(\varphi_t * a) = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t * a).$$

- For $S \in \mathcal{L}(\mathcal{A}^2)$,

$$\Delta(\varphi_t * S) = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t * S) := \left(\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t\right) * S$$

-

$$\varphi_t * S = \varphi_1 * S - \int_t^1 \frac{\mathrm{d}}{\mathrm{d}y}(\varphi_y * S)\ dy$$

Radial Toeplitz algebra

- (DM 2023,2024) radial operators $:= \mathcal{L}(\mathcal{F}^2)^{U(n)}$

$$\mathfrak{T}(L^\infty)^{U(n)} = \mathfrak{T}(L^\infty) \cap \mathcal{L}(\mathcal{F}^2)^{U(n)} = \overline{\{ T_a \mid a \in L^\infty(\mathbb{B}^n) \text{ is radial } \}} = \overline{D_\Delta \cap \mathcal{L}(\mathcal{F}^2)^{U(n)}}$$

- Let $\Gamma : \mathcal{L}(\mathcal{F}^2)^{U(n)} \to \ell^\infty(\mathbb{N}_0)$ be the spectral map, and d_Δ the image of $D_\Delta \cap \mathcal{L}(\mathcal{F}^2)^{U(n)}$ under Γ.

$$d_\Delta = \left\{ \{x_k\} \in \ell^\infty(\mathbb{N}_0) \mid \ \exists C > 0 \text{ s.t. } |\Delta^2 x_k| \leq \frac{C}{k+1} \forall k \right\}, \quad \Delta x_k = x_{k+1} - x_k.$$

$\overline{d_\Delta} = \mathcal{C}_{b,u}(\mathbb{N}_0, \rho)$

- For $\sigma \in \mathcal{C}_{b,u}(\mathbb{N}_0, \rho)$, extension f_σ^+ of σ to $\mathbb{R}_+$:

$$f_\sigma^+(x) = \sigma(k) + (\sigma(k+1) - \sigma(k))\frac{\sqrt{x} - \sqrt{k}}{\sqrt{k+1} - \sqrt{k}}.$$

- Define $f_\sigma \in \mathcal{C}_{b,u}(\mathbb{R})$ by

$$f_\sigma(x) = f_\sigma^+(x^2); \quad x \in \mathbb{R}.$$

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- Define $f_\sigma \in \mathcal{C}_{b,u}(\mathbb{R})$ by

$$f_\sigma(x) = f_\sigma^+(x^2); \quad x \in \mathbb{R}.$$

- $\mathcal{C}_{b,u}(\mathbb{N}_0, \rho) \to \mathcal{C}_{b,u}(\mathbb{R}), \ \sigma \mapsto f_\sigma$ is an isometry.
- $\mathcal{C}_{b,u}(\mathbb{R}) \to \mathcal{C}_{b,u}(\mathbb{N}_0, \rho), \ f \mapsto \sigma_f$, where $\sigma_f(k) = f(\sqrt{k}), \ n \in \mathbb{N}_0$, is a contraction.
- By MVT for divided differences, if f has a bounded sencond derivative then f_σ is in d_Δ. And such functions are dense in $\mathcal{C}_{b,u}(\mathbb{R})$.

Thank you!

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